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Lectures on minimal models

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INTRODUCTION

Algebraic topology has, classically, meant the study of algebraic invariants associated with topological spaces. These invariants (homology, homotopy,...) are normally not "geometric" in the sense that one cannot recover a space from them.

Although there is still no satisfactory algebraic description of homotopy theory (over $\mathbb{Z}$), the rational homotopy theory of Quillen and Sullivan is a practical and complete solution - if one is willing to forget torsion. Here one models the homotopy category by the category of commutative graded differential algebras (c.g.d.a.'s). Then to each c.g.d.a. one associates a "minimal model" with the property that if two c.g.d.a.'s are connected by a homomorphism which is an isomorphism of cohomology then the minimal models are isomorphic.

The process space $\to$ c.g.d.a. $\to$ minimal model gives the minimal model of a space. Its isomorphism class is an invariant of the weak homotopy type of the space, S. Moreover, if S is a 1-connected CW complex of finite type then from the model one can recover a space $S_{\mathbb{Q}}$ and a continuous map $S \to S_{\mathbb{Q}}$ which induces isomorphisms $\pi_i(S) \otimes \mathbb{Q} \cong \pi_i(S_{\mathbb{Q}})$.

Minimal models have proved to be a powerful tool in the solution of geometric problems. While the fact that one can indeed recover $S_{\mathbb{Q}}$ from the model is undoubtedly the philosophic reason for the power of the machine, this fact plays little direct role in the applications. Rather the two key ingredients turn out to be:

(a) A detailed understanding of the algebraic behaviour of the models, and

(b) A dictionary from classical topological invariants to invariants of the models.

My aim in these notes has been to provide a self-contained reference for many of the basic theorems needed for (a) and (b), in which complete, formal proofs were given down to the last technical detail. I have also tried to make the hypotheses as weak as possible and the conclusions as strong as possible. While this approach tends to make for difficult reading, it does (or so I hope!) result in a safely quoteable source for those whose main interest is the applications.

For the sake of completeness I have also included (with proofs!) many well known results and definitions (eg. simplicial sets in chap. 12, $\pi_1(M)$-modules in chap. 16 and Serre fibrations in chap. 19). In fact, the only prerequisite is some multilinear algebra and a little basic topology.

The material presented here divides naturally into three parts. The first (chaps. 1 to 11) is pure differential algebra: suppose

$$\eta : (B,d_B) \to (E,d_E)$$

is a homomorphism of c.g.d.a.'s (over a field k of characteristic zero). Assume $H^0(B) = H^0(E) = k$, and B is augmented.

Then there is a commutative diagram of c.g.d.a. homomorphisms

$$\begin{array}{ccccc} & & (E,d_E) & & \\ & \overset{\eta}{\nearrow} & \uparrow \psi & & \\ (B,d_B) & \longrightarrow & (B \otimes \Lambda X, d) & \longrightarrow & (\Lambda X, d_A) \end{array}$$

in which:

i) ψ^* is an isomorphism.

ii) ΛX is the free commutative graded algebra over the graded space X

iii) A certain "nilpotence-type" condition and a certain minimality condition (cf. chap. 1) are satisfied by d.

Moreover the bottom row is uniquely determined (up to isomorphism).

The diagram above is called the <u>minimal model</u> for η (cf. chap. 6). When $B = k$ we have simply

$$\psi : (\Lambda X, d_A) \to (E, d_E) \quad ;$$

it is called the minimal model for (E, d_E).

The second part of the theory is a functor $M \rightsquigarrow (A(M), d)$ from topological spaces to c.g.d.a.'s (over k) such that $H(A(M))$ is naturally isomorphic with the singular cohomology $H(M\ ;\ k)$. This is described in chaps. 13 to 15. The minimal model of $(A(M), d)$ is called the <u>minimal model</u> for M.

The third part is the study of fibrations (chaps. 16 to 20). Suppose $F \xrightarrow{j} E \xrightarrow{\pi} B$ is a Serre fibration in which F, E, B are path connected. Then we can form the model of $A(\pi) : A(B) \to A(E)$, obtaining the commutative diagram:

$$\begin{array}{ccccc} A(B) & \xrightarrow{A(\pi)} & A(E) & \xrightarrow{A(j)} & A(F) \\ \| & & \uparrow \psi & & \uparrow \alpha \\ A(B) & \longrightarrow & A(B) \otimes \Lambda X & \longrightarrow & \Lambda X \end{array}$$

in which ψ^* is an isomorphism. The fundamental theorem of this part reads

20.3. - <u>*Theorem*</u>. Assume that

i) Either $H(B\ ;\ k)$ or $H(F\ ;\ k)$ has finite type.

ii) $\pi_1(B)$ acts nilpotently in each $H^p(F\ ;\ k)$.

Then α^* is an isomorphism, and so $\alpha : \Lambda X \to A(F)$ is the minimal model for F.

This theorem was proved first by P. Grivel [G] in the case B is 1-connected. Another proof was given independently a little later by J.C. Thomas (unpublished), again for the case B is 1-connected. The proof given in these notes follows the general idea of Grivel's proof, but the technicalities are substantially more complex. In particular, heavy use is made of the notion of "local system over a simplicial set" (chap. 12) which is a simplicial analogue of a sheaf.

Let $\Lambda X \to A(M)$ be the minimal model (over $\mathbb{Q}$) of a path connected space M. There are obvious linear maps

$$X^p \to \mathrm{Hom}_{\mathbb{Z}}(\pi_p(M)\ ;\ \mathbb{Q}), \qquad p \geq 2.$$

Using theorem 20.3 it is easy to deduce the

Theorem. - Assume that

i) Each $\pi_p(M) \otimes \mathbb{Q}$ is a nilpotent finite dimensional $\pi_1(M)$ module for $p \geq 2$).

ii) The minimal model for $K(\pi_1(M)\ ;\ 1)$ has generators only in degree 1.

Then the linear maps $X^p \to \mathrm{Hom}_Z(\pi_p(M)\ ;\ \mathbb{Q})$, $p \geq 2$, are isomorphisms.

I had originally planned to include this and other applications, but ran out of time. They will appear elsewhere.

The theory of minimal models is due to Dennis Sullivan, and his paper "Infinitesimal Computations in Topology" [S] is the fundamental work on the subject. Indeed the first two parts of these notes (chaps. 1 to 11 and 13 to 15) follow [S] very closely.

The reader who makes this comparison will discover that aside from the occasional modification in the assertions of [S] I have frequently merely expanded the ideas there into formal proofs. (One exception is "de Rham's theorem" in chap. 14 whose proof is based on that of Chris Watkiss [W]; another version of this proof is given by Cartan [C]. Other proofs abound in the literature.) Of course the overlap of these notes with [S] covers only part of [S]. I haven't touched solvable models, let alone the latter half of [S].

Another approach to minimal models is via localizations and Postnikov towers. If M is a nilpotent space it can be localized to produce a rational space $M_{\mathbb{Q}}$. The data which define the Postnikov decomposition of $M_{\mathbb{Q}}$ are exactly the data which define the minimal model of M, and so it follows that the minimal model of M determines its rational homotopy type. The theorem above on homotopy groups follows at once, at least for nilpotent spaces. This approach is that of Friedlander et al. [F] and Lehmann [L_2]. The resumé by Lehmann [L_1] is particularly elegant and readable.

A different approach is taken by Bousfield and Gugenheim [B-G] who provide a complete exposition in the context of the closed model categories of Quillen. Other expositions (eg. [W-T]) are also available.

At least two other algebraic categories have been successfully used to model rational homotopy theory: the iterated integrals of Chen [Ch] and the category of graded differential Lie algebras. In the latter category the notion of minimal model was introduced by Baues and Lemaire [B-L].

The recent book of Tanré [Ta] provides a clear description of the relation between these categories and goes very much further than the present notes in describing topological invariants in terms of the model.

These notes are a greatly expanded version of lectures I gave at Lille in 1976 and 1977 in the seminar on algebraic topology and differential geometry. They first appeared in 1977 in the Publications Internes of the U.E.R. de Mathématiques, Université de Lille I and are presented here unchanged, except for changes to the introduction.

The seminar discussions were, naturally, enormously helpful - I want particularly to mention Daniel Lehmann and Chris Watkiss. My thanks also go to Mmes Tatti and Bérat for their careful typing of the manuscript, and to the Université de Lille I, whose hospitality made the whole thing possible. It is a great pleasure to be able, now, to say thank you to my Lille colleagues not only for their hospitality that year, but for all the subsequent years as well.

Finally, I should like to take this opportunity to express my warm gratitude to my teacher, friend and colleague Werner Greub from whom I first learned about commutative graded differential algebras and Koszul complexes.

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TABLE OF CONTENTS