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Mark Pollicott & Benedict Sewell

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75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

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AN UPPER BOUND ON THE DIMENSION OF THE RAUZY GASKET

BY MARK POLLICOTT & BENEDICT SEWELL

ABSTRACT. — In this note, we give an elementary proof that the Hausdorff dimension of the Rauzy gasket is at most 1.7415, improving upon results of Avila–Hubert–Skripchenko and Fougeron.

RÉSUMÉ (*Une borne supérieure de la dimension de la baderne de Rauzy*). — Nous donnons une démonstration élémentaire que la dimension Hausdorff de la baderne de Rauzy est d'au plus 1,7415, améliorant ainsi les résultats d'Avila-Hubert-Skripchenko et Fougeron.

1. Introduction

The Rauzy Gasket $\mathcal{G}$ is a compact subset of the standard 2-simplex, $\Delta = \{(x, y, z) \ : \ x, y, z \geq 0, \ x + y + z = 1\}$. It plays the role of an exceptional set in the theory of interval exchange transformations and other settings and is

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MARK POLLICOTT, Zeeman Building, Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom • *E-mail :* mpollic@maths.warwick.ac.uk
BENEDICT SEWELL, Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13–15, Budapest 1053, Hungary • *E-mail :* sewell@renyi.hu

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the limit set of the iterated function scheme for the projectivised linear maps $T_1, T_2, T_3 : \Delta \to \Delta$, defined by

$$T_1(x,y,z) = \left(\frac{1}{2-x}, \frac{y}{2-x}, \frac{z}{2-x}\right),$$
$$T_2(x,y,z) = \left(\frac{x}{2-y}, \frac{1}{2-y}, \frac{z}{2-y}\right),$$
$$T_3(x,y,z) = \left(\frac{x}{2-z}, \frac{y}{2-z}, \frac{1}{2-z}\right);$$

i.e., $\mathcal{G}$ is the smallest non-trivial closed set such that $\mathcal{G} = \bigcup_{j=1}^{3} T_j(\mathcal{G})$.

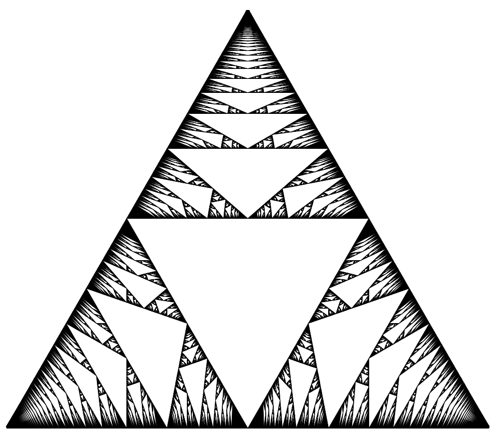

FIGURE 1.1. The Rauzy gasket

The gasket has an interesting history, appearing for the first time in 1991 in the work of Arnoux and Rauzy [1], in the context of interval exchange transformations, where it was conjectured that $\mathrm{Leb}(\mathcal{G}) = 0$. The gasket was rediscovered by Levitt in 1993 [12], in a paper that also included a proof (due to Yoccoz) that $\mathrm{Leb}(\mathcal{G}) = 0$. The gasket $\mathcal{G}$ emerged for a third time in the work of De Leo and Dynnikov [11]; this time in the context of Novikov's theory of magnetic induction on monocrystals (see [4] for the dichotomy between this and [1]). They gave an alternative proof that $\mathrm{Leb}(\mathcal{G}) = 0$ and proposed the stronger result $\dim_H(\mathcal{G}) < 2$. Novikov and Maltsev [13] also conjectured the stronger bound $\dim_H(\mathcal{G}) < 2$, which was rigorously established by Avila, Hubert and Skripchenko [3]. Empirical estimates in [11] suggest $\dim_H(\mathcal{G}) \approx 1.72$, and a lower bound was shown in [9]. Lastly, Fougeron used semiflows and thermodynamic techniques to show $\dim_H(\mathcal{G}) < 1.825$ [7]. Using completely elementary methods, we show the following improved upper bound.

THEOREM 1.1. — $\dim_H(\mathcal{G}) \leq 1.7415$.

The Rauzy Gasket has a number of interesting recent applications. Gamburd, Magee and Ronan [8] showed asymptotic estimates for integer solutions of the Markov–Hurwitz equations featuring $\dim_H(\mathcal{G})$. Hubert and Paris-Romaskevich in [10] considered triangular tiling billiards, modelling refraction in crystals. The gasket $\mathcal{G}$ parameterises triangles admitting trajectories that escape non-linearly to infinity and closed orbits that approximate fractal-like sets.

In Section 2, we give the technical result that leads to the bound in Theorem 1.1. This is formulated in terms of certain infinite matrices. In Section 3, we give elementary preliminary bounds on the area and diameter of small triangles given as the images of Δ under compositions of the maps T_1, T_2 and T_3. In Section 4, we use these to obtain a bound for the dimension, provided an associated sequence of real numbers X_n converges to zero. In Sections 5 and 6 we present the core of the proof. In Section 5, we use the estimates from Section 3 to bound X_n in terms of expressions satisfying an iterative relation. In Section 6, we use the renewal theorem to deduce that $X_n \to 0$ under the hypotheses of Theorem 2.8. Finally, in Section 7, we apply Theorem 2.8 empirically to deduce the bound in Theorem 1.1.

A fuller account appears in [14].

2. A formal statement

The bound in Theorem 1.1 is a special case of a decreasing sequence of upper bounds, indexed by a parameter $m \in \mathbb{N}_{\geq 2}$. Each bound can be described using powers of an infinite matrix.

DEFINITION 2.1 (An index set). — Let $\mathcal{V} = \bigcup_{k=1}^{m-1} \mathcal{V}_k$ denote the finite set where, for $k < m$,

$$\begin{aligned}\mathcal{V}_k &:= \{1\}^k \times \{2\} \times \{1,2,3\}^{m-k} \\ &= \left\{(1^k, 2, v_{k+2}, \ldots, v_{m+1}) \ : \ v_{k+2}, \ldots, v_{m+1} \in \{1,2,3\}\right\},\end{aligned}$$

where we denote, e.g. $1^k = \overbrace{1, \ldots, 1}^{k}$; i.e. $\mathcal{V}$ is the family of strings of length $m+1$ beginning with a sequence of 1s of length k, followed by a 2, then a sequence of 1s, 2s and 3s of length $m-k$.

REMARK 2.2. — The elements of $\mathcal{V}$ represent orbits in $\{1,2,3\}^{m+1}$ under the natural action of the dihedral group, excluding the orbits of $(1^m, 2)$ and (1^{m+1}).

More specifically, any two words $\underline{i}, \underline{v} \in \{1,2,3\}^{m+1}$ will be considered equivalent (written $\underline{i} \sim \underline{v}$) if there is some permutation π of $\{1,2,3\}$ such that $\pi(i_j) = v_j$ for all $j = 1, \ldots, m+1$. For example, if $i_1 \neq i_2$, (i_1, i_2^m) is equivalent to $(1, 2^m) =: \circledast$, which we consider as a distinguished element of $\mathcal{V}$.