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THEORY OF $\mathcal{H}_p$-SPACES FOR CONTINUOUS FILTRATIONS IN VON NEUMANN ALGEBRAS

Marius Junge, Mathilde Perrin

***Abstract.* —** We introduce Hardy spaces for martingales with respect to continuous filtration for von Neumann algebras. In particular we prove the analogues of the Burkholder-Gundy and Burkholder-Rosenthal inequalities in this setting. The usual arguments using stopping times in the commutative case are replaced by tools from noncommutative function theory and allow us to obtain the analogue of the Feffermann-Stein duality and prove a noncommutative Davis decomposition.

***Résumé* (Théorie des espaces $\mathcal{H}_p$ pour des filtrations continues dans des algèbres de von Neumann)**

Nous introduisons des espaces de Hardy pour des martingales relatives à des filtrations continues d'algèbres de von Neumann. Nous démontrons en particulier les inégalités de Burkholder-Gundy et de Burkholder-Rosenthal dans ce cadre. Les arguments usuels basés sur des temps d'arrêt dans le cas commutatif sont remplacés par des outils de la théorie des fonctions non commutatives, qui nous permettent d'obtenir l'analogue de la dualité de Fefferman-Stein et de prouver une décomposition de Davis non commutative.

CONTENTS

CHAPTER 1

INTRODUCTION

The theory of stochastic integrals and martingales with continuous time is a well-known theory with many applications. Quantum stochastic calculus is also well developed with applications reaching into fields such as quantum optics. In the setting of von Neumann algebras, many classical martingale inequalities have been reformulated for noncommutative martingales with respect to discrete filtrations, see *e.g.* [**40**], [**27**], [**21**], [**30**]. The aim of this paper is to study martingales with respect to continuous filtrations in von Neumann algebras. Our long term goal is to develop a satisfactory theory for semimartingales, including the convergence of the stochastic integrals. In the noncommutative setting, we cannot construct the stochastic integrals pathwise as in [**9**]. It is unimaginable to consider the path of a process of operators in a von Neumann algebra. However, it is well-known that in the classical case, the convergence of the stochastic integrals is closely related to the existence of the quadratic variation bracket $[.,.]$ via the formula

$$X_tY_t = \int^t X_{s^-}\,\mathrm{d}Y_s + \int^t Y_{s^-}\,\mathrm{d}X_s + [X,Y]_t.$$

Here the quadratic variation bracket can be characterized as the limit in probability of the following dyadic square functions

$$[X,Y]_t = X_0Y_0 + \lim_{n\to\infty}\sum_{k=0}^{2^n-1}(X_{t\frac{k+1}{2^n}} - X_{t\frac{k}{2^n}})(Y_{t\frac{k+1}{2^n}} - Y_{t\frac{k}{2^n}}).$$

Hence we will first study this quadratic variation bracket in the setting of von Neumann algebras, and then deal with stochastic integrals in a forthcoming paper based on the theory developed here. More precisely, we will focus on the $L_{\frac{1}{2}p}$-norm of this bracket by considering the Hardy spaces H_p defined in the classical case by the norm

$$\|x\|_{H_p} = \big\|[x,x]\big\|_{\frac{1}{2}p}^{\frac{1}{2}}.$$

This paper develops a theory of the Hardy spaces of noncommutative martingales with respect to a continuous filtration. One fundamental application is an interpolation theory for these noncommutative function spaces which has already found applications in the theory of semigroups (see *e.g.* [**22**]).

Let us consider a von Neumann algebra $\mathcal{M}$. For simplicity, we assume that $\mathcal{M}$ is finite and equipped with a normal faithful normalized trace τ. Fortunately, the theory of noncommutative H_p-spaces is now very well understood in the discrete setting, *i.e.*, when dealing with an increasing sequence $(\mathcal{M}_n)_{n\geq 0}$ of von Neumann subalgebras of $\mathcal{M}$, whose union is weak*-dense in $\mathcal{M}$. We consider the associated conditional expectations $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$. In the noncommutative setting it is well-known that we always encounter two different objects, the row and column versions of the Hardy spaces:

$$\|x\|_{H_p^c} = \Big\|\Big(\sum_n |d_n(x)|^2\Big)^{\frac{1}{2}}\Big\|_p \quad\text{and}\quad \|x\|_{H_p^r} = \Big\|\Big(\sum_n |d_n(x^*)|^2\Big)^{\frac{1}{2}}\Big\|_p,$$

where $d_n(x) = \mathcal{E}_n(x) - \mathcal{E}_{n-1}(x)$. Here $\|x\|_p = (\tau(|x|^p))^{1/p}$ refers to the norm in the noncommutative L_p-space. The noncommutative Burkholder-Gundy inequalities from [**40**] say that with equivalent norms for$1 < p < \infty$,

$$L_p(\mathcal{M}) = H_p, \tag{1.0.1}$$

where the H_p-space is defined by

$$H_p = \begin{cases} H_p^c + H_p^r & \text{for} \quad 1 \leq p < 2, \\ H_p^c \cap H_p^r & \text{for} \quad 2 \leq p < \infty. \end{cases}$$

Following the commutative theory, we should expect to define the bracket $[x, x]$ for a martingale x and then define

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \big\|[x, x]\big\|_{\frac{1}{2}p}^{\frac{1}{2}} \quad \text{and} \quad \|x\|_{\widehat{\mathcal{H}}_p^r} = \big\|[x^*, x^*]\big\|_{\frac{1}{2}p}^{\frac{1}{2}}.$$

Armed with the definition we may then attempt to prove (1.0.1) for a continuous filtration $(\mathcal{M}_t)_{t\geq 0}$. For simplicity, we assume that the continuous parameter set is given by the interval $[0, 1]$. We define a candidate for the noncommutative bracket following a nonstandard analysis approach. For a finite partition $\sigma = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of the interval $[0, 1]$ and $x \in \mathcal{M}$, we consider the finite bracket

$$[x, x]_\sigma = \sum_{t\in\sigma} \big|d_t^\sigma(x)\big|^2,$$

where $d_t^\sigma(x) = \mathcal{E}_t(x) - \mathcal{E}_{t^-(\sigma)}(x)$. Then for $p > 2$, (1.0.1) gives an *a priori* bound

$$\big\|[x, x]_\sigma]\big\|_{\frac{1}{2}p}^{\frac{1}{2}} \leq \alpha_p \|x\|_p.$$

Hence, for a fixed ultrafilter $\mathcal{U}$ refining the general net of finite partitions of $[0, 1]$, we may simply define

$$[x, x]_\mathcal{U} = w\text{-}L_{\frac{1}{2}p}\text{-}\lim_{\sigma,\mathcal{U}} [x, x]_\sigma.$$

In fact, in nonstandard analysis, the weak-limit corresponds to the standard part and is known to coincide with the classical definition of the bracket for commutative martingales. However, the norm is only lower semi-continuous with respect to the weak topology and we should not expect Burkholder/Gundy inequalities for continuous filtrations to be a simple consequence of the discrete theory of H_p-spaces. Yet, using the crucial observation that the $L_{\frac{1}{2}p}$-norms of the discrete brackets $[x, x]_\sigma$ are monotonous up to a constant, we may show the following result.

Theorem 1.0.1. — *Let $1 \leq p < \infty$ and $x \in \mathcal{M}$. Then*

$$\big\|[x, x]_\mathcal{U}\big\|_{\frac{1}{2}p} \simeq \lim_{\sigma,\mathcal{U}} \big\|[x, x]_\sigma\big\|_{\frac{1}{2}p} \simeq \begin{cases} \sup_\sigma \big\|[x, x]_\sigma\big\|_{\frac{1}{2}p} & \textit{for} \quad 1 \leq p < 2, \\ \inf_\sigma \big\|[x, x]_\sigma\big\|_{\frac{1}{2}p} & \textit{for} \quad 2 \leq p < \infty. \end{cases}$$

In particular, this implies that the $L_{\frac{1}{2}p}$-norm of the bracket $[x, x]_\mathcal{U}$ does not depend on the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm. We will discuss the independence of the bracket $[x, x]_\mathcal{U}$ itself from the choice of $\mathcal{U}$ in a forthcoming paper. Hence for $1 \leq p < \infty$ and $x \in \mathcal{M}$ we define the norms

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \big\|[x, x]_\mathcal{U}\big\|_{\frac{1}{2}p}^{\frac{1}{2}} \quad \text{and} \quad \|x\|_{\mathcal{H}_p^c} = \lim_{\sigma,\mathcal{U}} \big\|[x, x]_\sigma\big\|_{\frac{1}{2}p}^{\frac{1}{2}} = \lim_{\sigma,\mathcal{U}} \|x\|_{H_p^c(\sigma)}.$$

We denote by $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ respectively the corresponding completions. Using theorem 1.0.1 we may show that actually

$$\widehat{\mathcal{H}}_p^c = \mathcal{H}_p^c \quad \text{with equivalent norms for } 1 \leq p < \infty. \tag{1.0.2}$$

Hence this defines a good candidate for the Hardy space of noncommutative martingales with respect to the continuous filtration $(\mathcal{M}_t)_{0\leq t\leq 1}$. We now want to establish for this space the analogues of many well-known results in the discrete setting. For doing this, we will use the definition of the space $\mathcal{H}_p^c$, which will be more practical to work with. In particular, we may embed $\mathcal{H}_p^c$ into some ultraproduct space, which has an L_p-module structure and a p-equiintegrability property. This