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AUTOMATIC CONTINUED FRACTIONS ARE TRANSCENDENTAL OR QUADRATIC

BY YANN BUGEAUD

ABSTRACT. – We establish new combinatorial transcendence criteria for continued fraction expansions. Let $\alpha = [0; a_1, a_2, \dots]$ be an algebraic number of degree at least three. One of our criteria implies that the sequence of partial quotients $(a_\ell)_{\ell \geq 1}$ of α is not 'too simple' (in a suitable sense) and cannot be generated by a finite automaton.

RÉSUMÉ. – Nous établissons de nouveaux critères combinatoires de transcendance pour des développements en fraction continue. Soit $\alpha = [0; a_1, a_2, \dots]$ un nombre algébrique de degré au moins égal à trois. L'un de nos critères entraîne que la suite $(a_\ell)_{\ell \geq 1}$ des quotients partiels de α n'est pas trop simple (en un certain sens) et ne peut pas être engendrée par un automate fini.

1. Introduction and results

A well-known open question in Diophantine approximation asks whether the continued fraction expansion of an irrational algebraic number α either is ultimately periodic (this is the case if, and only if, α is a quadratic irrational), or contains arbitrarily large partial quotients. As a preliminary step towards its resolution, several transcendence criteria for continued fraction expansions have been established recently [1, 4, 5, 9] (we refer the reader to these papers for references to earlier works, which include [23, 14, 27, 12]) by means of a deep tool from Diophantine approximation, namely the Schmidt Subspace Theorem (see Theorem 2.1 below). In the present note, we show how a slight modification of their proofs allows us to considerably improve two of these criteria. We begin by pointing out two important consequences of one of our new criteria. Thus, we solve two problems addressed and discussed in [1] and we establish for continued fraction expansions of algebraic numbers the analogues of the results of [3] on the expansions of algebraic numbers to an integer base.

Throughout this note, $\mathcal{A}$ denotes a finite or infinite set, called the alphabet. We identify a sequence $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ of elements from $\mathcal{A}$ with the infinite word $a_1 a_2 \dots a_\ell \dots$, as well denoted by $\mathbf{a}$. This should not cause any confusion.

For $n \geq 1$, we denote by $p(n, \mathbf{a})$ the number of distinct blocks of n consecutive letters occurring in the word $\mathbf{a}$, that is,

$$p(n, \mathbf{a}) := \mathrm{Card}\{a_{\ell+1} \dots a_{\ell+n} : \ell \geq 0\}.$$

The function $n \mapsto p(n, \mathbf{a})$ is called the *complexity function* of $\mathbf{a}$. A well-known result of Morse and Hedlund [24, 25] asserts that $p(n, \mathbf{a}) \geq n + 1$ for $n \geq 1$, unless $\mathbf{a}$ is ultimately periodic (in which case there exists a constant C such that $p(n, \mathbf{a}) \leq C$ for $n \geq 1$).

Our first result asserts that the complexity function of the sequence of partial quotients $(a_\ell)_{\ell \geq 1}$ of an algebraic number

$$[0; a_1, a_2, \dots, a_\ell, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

of degree at least three cannot increase too slowly.

THEOREM 1.1. – *Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of positive integers which is not ultimately periodic. If the real number*

$$[0; a_1, a_2, \dots, a_\ell, \dots]$$

is algebraic, then

$$\lim_{n \to +\infty} \frac{p(n, \mathbf{a})}{n} = +\infty. \tag{1.1}$$

Theorem 1.1 improves Theorem 7 from [12] and Theorem 4 from [1], where

$$\lim_{n \to +\infty} p(n, \mathbf{a}) - n = +\infty$$

was proved instead of (1.1). This gives a positive answer to Problem 3 of [1] (we have chosen here a different formulation).

An infinite sequence $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ is an automatic sequence if it can be generated by a finite automaton, that is, if there exists an integer $k \geq 2$ such that a_ℓ is a finite-state function of the representation of ℓ in base k, for every $\ell \geq 1$. We refer the reader to [13] for a more precise definition and examples of automatic sequences. Let $b \geq 2$ be an integer. In 1968, Cobham [19] asked whether a real number whose b-ary expansion can be generated by a finite automaton is always either rational or transcendental. A positive answer to Cobham's question was recently given in [3]. We addressed in [1] the analogous question for continued fraction expansions. Since the complexity function of every automatic sequence $\mathbf{a}$ satisfies $p(n, \mathbf{a}) = O(n)$ (this was proved by Cobham [20] in 1972), Theorem 1.1 implies straightforwardly a negative answer to Problem 1 of [1].

THEOREM 1.2. – *The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a finite automaton.*

The proofs of Theorems 1.1 and 1.2 rest ultimately on a combinatorial transcendence criterion established by means of the Schmidt Subspace Theorem. This is also the case for the similar results about expansions of irrational algebraic numbers to an integer base, see [3, 10].

Before stating our criteria, we introduce some notation. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. We denote the mirror image of a finite word $W := a_1 \dots a_\ell$ by $\overline{W} := a_\ell \dots a_1$. In particular, W is a palindrome if, and only if, $W = \overline{W}$.

Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition $(*)$ if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that:

(i) For every $n \geq 1$, either the word $W_n U_n V_n U_n$ or the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;
(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded from above;
(iii) The sequence $(|W_n|/|U_n|)_{n\geq 1}$ is bounded from above;
(iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Equivalently, the word $\mathbf{a}$ satisfies Condition $(*)$ if there exists a positive real number ε such that, for arbitrarily large integers N, the prefix $a_1 a_2 \dots a_N$ of $\mathbf{a}$ contains two disjoint occurrences of a word of length $[\varepsilon N]$ or it contains a word U of length $[\varepsilon N]$ and its mirror image $\overline{U}$, provided that U and $\overline{U}$ do not overlap. Here and below, $[\cdot]$ denotes the integer part function.

We summarize our two new combinatorial transcendence criteria in the following theorem.

THEOREM 1.3. – *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \dots, a_\ell, \dots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathbf{a}$ satisfies Condition $(*)$, then α is transcendental.

Theorem 1.3 is the disjoint union of two transcendence criteria. A first one applies to stammering continued fractions, where the terminology ‘stammering’ means that in (i) the word $W_n U_n V_n U_n$ is a prefix of the word $\mathbf{a}$ for infinitely many n; see Theorem 3.1. A second one is concerned with quasi-palindromic continued fractions, where the terminology ‘quasi-palindromic’ means that in (i) the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$ for infinitely many n; see Theorem 5.1. The condition that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ has to be bounded is not very restrictive, since it is satisfied by almost all real numbers (in the sense of the Lebesgue measure). Furthermore, it is clearly satisfied when $(a_\ell)_{\ell\geq 1}$ is bounded. Note that this condition can be removed if $\mathbf{a}$ begins with arbitrarily large squares $U_n U_n$ (Theorem 2.1 from [9]) or with arbitrarily large palindromes $U_n \overline{U}_n$ (Theorem 2.1 from [5]).

Theorem 1.3 encompasses all the combinatorial transcendence criteria for continued fraction expansions established in [1, 4, 5, 9] under the assumption that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded.

Let $\mathbf{a}$ be a sequence of positive integers. If there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that $\limsup_{n\to+\infty} |W_n|/|U_n|$ is sufficiently small and $\mathbf{a}$ satisfies Condition $(*)$, then the transcendence of $[0; a_1, a_2, \dots]$ was already proved in [1, 9, 5]. The novelty in Theorem 1.3 is that we allow $|W_n|$ to be large, provided however that the quotients $|W_n|/|U_n|$ remain bounded independently of n. This is crucial for the proofs of Theorems 1.1 and 1.2.

At present, we do not know any transcendence criterion involving palindromes for expansions to integer bases; however, see [2].

We end this section with an application of Theorem 3.1 to quasi-periodic continued fractions.

THEOREM 1.4. – *Consider the quasi-periodic continued fraction*

$$\alpha = [0; a_1, \ldots, a_{n_0-1}, \underbrace{a_{n_0}, \ldots, a_{n_0+r_0-1}}_{\lambda_0}, \underbrace{a_{n_1}, \ldots, a_{n_1+r_1-1}}_{\lambda_1}, \ldots],$$

where the notation means that $n_{k+1} = n_k + \lambda_k r_k$ and the λ's indicate the number of times a block of partial quotients is repeated. Let $(p_\ell/q_\ell)_{\ell\ge 1}$ denote the sequence of convergents to α. Assume that the sequence $(q_\ell^{1/\ell})_{\ell\ge 1}$ is bounded. If the sequence $(a_\ell)_{\ell\ge 1}$ is not ultimately periodic and

$$\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} > 1, \tag{1.2}$$

then the real number α is transcendental.

Theorem 1.4 improves Theorem 3.4 from [4], where, instead of the Assumption (1.2), the stronger condition $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 2$ was required.

2. Auxiliary results

We gather below several classical results from the theory of continued fractions. Standard references include [26, 22, 29].

Let $\alpha := [0; a_1, a_2, \ldots]$ be a real irrational number. Set $p_{-1} = q_0 = 1$ and $q_{-1} = p_0 = 0$. For $\ell \ge 1$, the ℓ-th convergent to α is the rational number $p_\ell/q_\ell := [0; a_1, a_2, \ldots, a_\ell]$. Observe that

$$q_\ell = a_\ell q_{\ell-1} + q_{\ell-2}, \quad \ell \ge 1. \tag{2.1}$$

Furthermore, the sequence $(q_\ell)_{\ell\ge 1}$ is increasing and q_ℓ and $q_{\ell+1}$ are coprime for $\ell \ge 0$.

The theory of continued fraction implies that (see e.g., Theorem 164 of [22])

$$|q_\ell\alpha - p_\ell| < q_{\ell+1}^{-1}, \quad \text{for } \ell \ge 1, \tag{2.2}$$

and

$$q_{\ell+h} \ge q_\ell (\sqrt{2})^{h-1}, \quad \text{for } h, \ell \ge 1. \tag{2.3}$$

Indeed, an easy induction on h based on (2.1) proves (2.3) for every fixed value of $\ell \ge 1$.

Likewise, an induction based on (2.1) allows us to establish the mirror formula

$$\frac{q_{\ell-1}}{q_\ell} = [0; a_\ell, a_{\ell-1}, \ldots, a_1], \quad \ell \ge 1. \tag{2.4}$$

The main tool for the proof of Theorem 1.3 is the Schmidt Subspace Theorem.