ISSN 0012-9593

quatrième série - tome 47 fascicule 6 novembre-décembre 2014

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 47, 2014, p. 1027 à 1083

STATISTICAL PROPERTIES OF ONE-DIMENSIONAL MAPS UNDER WEAK HYPERBOLICITY ASSUMPTIONS

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ABSTRACT. – For a real or complex one-dimensional map satisfying a weak hyperbolicity assumption, we study the existence and statistical properties of physical measures, with respect to geometric reference measures. We also study geometric properties of these measures.

RÉSUMÉ. – Nous étudions l'existence et des propriétés statistiques des mesures physiques d'une application unidimensionnelle réelle ou complexe satisfaisant une hypothèse d'hyperbolicité faible, par rapport à une mesure de référence géométrique. Nous étudions aussi des propriétés géométriques de ces mesures.

1. Introduction

We study statistical properties of real and complex one-dimensional maps, under weak hyperbolicity assumptions. For such a map f we are interested in the existence and statistical properties of an invariant probability measure ν, supported on the Julia set of f, that is absolutely continuous with respect to a natural reference measure. The reference measure μ could be the Lebesgue measure on the phase space, or more generally a conformal measure supported on the Julia set. Such a measure ν, when ergodic, has the important property of being a *physical measure* with respect to μ. That is, for a subset E of the phase space that has positive measure with respect to μ, the measure ν describes the asymptotic distribution of each forward orbit of f starting at a point in E.

For maps that are uniformly hyperbolic on their Julia sets, the pioneering work of Sinaĭ, Ruelle, and Bowen [53, 4, 51] gives a satisfactory solution to these problems. See also [55] for an analysis closer to the approach here. However, a one-dimensional map with a critical point in its Julia set fails to be uniformly hyperbolic in a severe way. In order to control the

The first named author is partially supported by the Research Network on Low Dimensional Systems, PBCT/CONICYT, Chile. He gratefully acknowledges the hospitality of the University of Science and Technology of China and the National University of Singapore. The second named author is partially supported by a start-up grant from National University of Singapore (Grant No. R-146-000-128-133).

effect of critical points in the Julia set, people often assume strong expansion along the orbits of critical values. See for example [8, 41, 2, 43, 23, 57, 5] in the real setting, and [44, 15, 16] in the complex setting. See [3] for a broad view.

For smooth interval maps, Bruin, Luzzatto and van Strien gave mixing rates upper bounds closely related to the growth of derivatives at the critical values [5]. Our results reveal that, rather surprisingly, the mixing rates can be much faster than the growth of derivatives at critical values: an interval map f satisfying the *Large Derivatives* condition

$$\lim_{n\to\infty} |Df^n(v)| = \infty, \text{ for each critical value } v \text{ of } f \text{ in the Julia set}$$

together with other mild conditions, has a super-polynomially mixing absolutely continuous invariant measure.

In the complex setting we show a similar result for a non-renormalizable polynomial f. These are the first non-exponential upper bounds for mixing rates in the complex setting. For a general rational map f without parabolic cycles we show that the summability condition with exponent 1 is enough to guarantee the existence of a super-polynomially mixing absolutely continuous invariant measure.

We shall now state two results, one for the real case and another for the complex case, and make comparisons with previous results. In order to avoid technicalities, we state these results in a more restricted situation than what we are able to handle. See § 2.1 for a more general formulation of our results and for precisions.

Recall that given a continuous map f acting on a compact metric space X, an f-invariant Borel probability measure ν is called *(strongly) mixing* if for all $\varphi, \psi \in L^2(X, \nu)$,

$$\mathscr{C}_n(\varphi, \psi) := \int_X \varphi \circ f^n \psi d\nu - \int_X \varphi d\nu \int_X \psi d\nu \to 0$$

as $n \to \infty$. Given $\gamma > 0$, we say that ν is *polynomially mixing of exponent* γ if for each essentially bounded function φ and each Hölder continuous function ψ, there exists a constant $C(\varphi, \psi) > 0$ such that

$$|\mathscr{C}_n(\varphi, \psi)| \le C(\varphi, \psi) n^{-\gamma}, \text{ for all } n = 1, 2, \ldots$$

Moreover, we say that ν is *super-polynomially mixing* if for all $\gamma > 0$ it is polynomially mixing of exponent γ.

THEOREM I. – *Let X be a compact interval and let $f : X \to X$ be a topologically exact C^3 multimodal map with non-flat critical points, having only hyperbolic repelling periodic points. Assume that for each critical value v of f we have*

$$\lim_{n\to\infty} |Df^n(v)| = \infty.$$

Then f has a unique invariant probability measure that is absolutely continuous with respect to Lebesgue measure. Moreover, this invariant measure is super-polynomially mixing.

The topological exactness is assumed to obtain uniqueness and the mixing property of the absolutely continuous invariant measure. For an interval map as in the theorem, the existence of the absolutely continuous invariant measure was proved before in [7, 6], although the argument in this paper provides an alternative proof. As mentioned above, our result on mixing rates significantly strengthens the previous result [5], where super-polynomial mixing rates were only proved under the condition that for each $\alpha > 0$ and each critical value v of f, we have $|Df^n(v)|/n^\alpha \to \infty$. In fact, only assuming $\liminf_{n\to\infty} |Df^n(v)|$ sufficiently large, our methods provide a definite polynomial mixing rate.

We now state a result for a complex rational map f of degree at least two. Often the Lebesgue measure of the Julia set $J(f)$ of f is zero. So the Lebesgue measure cannot be used as a reference measure in general. Instead people often use a conformal measure on the Julia set as a reference measure. Following Sullivan [55], we use conformal measures of exponent $\mathrm{HD}(J(f))$ as geometric reference measures, where $\mathrm{HD}(J(f))$ denotes the Hausdorff dimension of $J(f)$.

Theorem II. – *Let f be either one of the following:*

1. *an at most finitely renormalizable polynomial of degree at least two, that has only hyperbolic periodic points, and such that for each critical value v of f in the Julia set,*
$$\lim_{n\to\infty} |Df^n(v)| = \infty;$$
2. *a complex rational map of degree at least two, without parabolic cycles, and such that for each critical value v of f in the Julia set,*
$$\sum_{n=1}^{\infty} \frac{1}{|Df^n(v)|} < \infty.$$

Then f has a unique conformal measure μ of exponent $\mathrm{HD}(J(f))$*; this measure is supported on the conical Julia set and its Hausdorff dimension is equal to* $\mathrm{HD}(J(f))$*. Furthermore, there is a unique invariant probability measure ν that is absolutely continuous with respect to μ, and the measure ν is super-polynomially mixing.*

Recall that for an integer $s \geq 1$, a complex polynomial f is *renormalizable of period s* if there are Jordan disks $U \Subset V$ such that the following hold:

- $f^s : U \to V$ is proper of degree at least two;
- the set $\{z \in U : f^{sn}(z) \in U \text{ for all } n = 1, 2, \ldots\}$ is a connected proper subset of $J(f)$;
- for each critical point c of f, there exists at most one $j \in \{0, 1, \ldots, s-1\}$ with $c \in f^j(U)$.

We say that f is *infinitely renormalizable* if there are infinitely many s for which f is renormalizable of period s.

For a complex polynomial f, hypothesis 1 of Theorem II is weaker than hypothesis 2.

Theorem II gives the first non-exponential mixing rates in the complex setting. As for the existence of the absolutely continuous invariant measure, this result gives a significant improvement of the previous result of Graczyk and Smirnov [16, Theorem 4]. Their result applies to a rational map f satisfying the following strong form of the summability condition, for a sufficiently small $\alpha \in (0, 1)$,

$$\sum_{n=1}^{\infty} \frac{n}{|Df^n(v)|^\alpha} < \infty, \text{ for every critical value } v \text{ of } f \text{ in } J(f). \tag{1}$$

For each $\alpha \in (0,1)$, the Fibonacci quadratic polynomial f_0 fails to satisfy this condition, although for every $\alpha > 0$

$$\sum_{n=1}^{\infty} \frac{1}{|Df_0^n(v)|^{\alpha}} < \infty, \text{ where } v \text{ is the finite critical value of } f_0,$$

see Remark 2.5. So Theorem II implies that the Fibonacci quadratic polynomial f_0 has a super-polynomially mixing absolutely continuous invariant measure.

REMARK 1.1. – In the proof of Theorems I and II we construct the absolutely continuous invariant measure by way of an inducing scheme with a super-polynomial tail estimate and some additional technical properties, see § 2.2. The results of [58] imply that this measure is super-polynomially mixing and that it satisfies the Central Limit Theorem for Hölder continuous observables. It also follows that the absolutely continuous invariant measure has other statistical properties, such as the Local Central Limit Theorem, and the Almost Sure Invariance Principle, see *e.g.,* [14, 35, 36, 56].

For a map f satisfying the hypotheses of Theorem I or Theorem II we show the density of the absolutely continuous invariant measure has the following regularity: if we denote by ℓ the maximal order of a critical point of f in the Julia set, then for each $p \in (0, \ell/(\ell-1))$ the invariant density belongs to the space L^p. We note that for each $p > \ell/(\ell-1)$ the invariant density does not belong to L^p, see Remark 2.17. In the real case the regularity of the invariant density was shown in [6, Main Theorem]; see also [43] for the case of unimodal maps satisfying a summability condition with a certain exponent. In the complex setting our result seems to be the first unconditional one. For rational maps satisfying a summability condition with a sufficiently small exponent, a similar result was shown in [16, Corollary 10.1] under an integrability assumption on the conformal measure μ that was first formulated in [44]. Actually, in the complex case we shall prove for each $\varepsilon > 0$ the following regularity of the conformal measure μ: for every sufficiently small $\delta > 0$ we have for every $x \in J(f)$,

$$\delta^{\mathrm{HD}(J(f))+\varepsilon} \le \mu(B(x,\delta)) \le \delta^{\mathrm{HD}(J(f))-\varepsilon}.$$

The lower bound is [28, Theorem 1], while the upper bound is new and implies the integrability condition for each exponent $\eta < \mathrm{HD}(J(f))$, see (3) in § 2.1.

Let us say a few words on our strategy. Prior to this work, it has been shown that a map satisfying the assumptions of Theorem I or of Theorem II has the following two expanding properties: "*expansion away from critical points*" and "*backward contraction*". Roughly speaking, the first property means that outside any given neighborhood of the critical points the map is uniformly hyperbolic; the second property means that a return domain to a ball of radius δ centered at a critical value is much smaller than δ. See § 2.1 for the precise definitions and references, as well as our "Main Theorem" stated for maps satisfying these two expanding properties.

In this paper, we provide a finer quantification of the expansion features of a map that satisfies the two properties stated above. Firstly, we show that the components of the preimages of a small ball intersecting the Julia set shrink *at least at a super-polynomial rate* (Theorem A in § 2.2.1). This unexpected result represents a significant improvement on the estimate of the same type in [16, Proposition 7.2], for rational maps satisfying the summability condition