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OPTIMAL INTEGRAL PINCHING RESULTS

BY Vincent BOUR AND Gilles CARRON

Abstract. – In this article, we generalize the classical Bochner-Weitzenböck theorem for manifolds satisfying an integral pinching on the curvature. We obtain the vanishing of Betti numbers under integral pinching assumptions on the curvature, and characterize the equality case. In particular, we reprove and extend to higher degrees and higher dimensions a number of integral pinching results obtained by M. Gursky for four-dimensional closed manifolds.

Résumé. – La formule de Bochner-Weitzenböck implique qu'une variété riemannienne compacte dont l'opérateur de courbure est strictement positif a tous ses nombres de Betti triviaux. Nous obtenons un tel résultat d'annulation sous des hypothèses de pincement intégral sur la courbure. Nos résultats sont optimaux et nous analysons les cas d'égalités. Il s'agit d'une extension à la dimension supérieure d'un résultat de M. Gursky.

1. Introduction

The Bochner method has led to important relations between the topology and the geometry of Riemannian manifolds (see [5] for instance). The original theorem of S. Bochner asserts that a closed n-dimensional Riemannian manifold with nonnegative Ricci curvature has a first Betti number smaller than n. The technique used by S. Bochner has been refined, and the result extended to Betti numbers of higher degrees and to various notions of positive curvature. For instance S. Gallot and D. Meyer proved in [24] that the Betti numbers of a closed n-dimensional manifold with nonnegative curvature operator must be smaller than those of the torus of dimension n. More precisely, they proved that if a closed Riemannian manifold (M^n, g) has a nonnegative curvature operator, i.e., if

$$\rho_g \leq \frac{1}{n(n-1)} R_g, \tag{1.1}$$

where $-\rho_g$ stands for the lowest eigenvalue of the traceless curvature operator and R_g is the scalar curvature of (M^n, g), then for all $1 \leq k \leq \frac{n}{2}$,

– either its kth Betti number $b_k(M^n)$ vanishes,

– or equality holds in (1.1), $1 \leq b_k \leq \binom{n}{k}$ and every harmonic k-form is parallel.

Recently, using the Ricci flow, C. Böhm and B. Wilking proved that a Riemannian manifold with positive curvature operator (i.e., which satisfies the strict inequality in (1.1)) is not only a homological sphere, but is in fact diffeomorphic to a spherical space form ([6]). A little while later, S. Brendle and R. Schoen proved that this is still true for manifolds with 1/4-pinched sectional curvature ([13, 12]).

In 1998, in his paper [25], M. Gursky obtained several Bochner's type theorems in dimension four. The striking fact in his work is that the assumption on the curvature is only required in an integral sense. He later refined part of his results in [26].

Our formulation of M. Gursky's results will be given in term of the Yamabe invariant

$$\mathrm{Y}(M,g) := \inf_{\substack{\varphi\in C_0^\infty(M)\\ \varphi\neq 0}} \frac{\int_M \left[\frac{4(n-1)}{n-2}|d\varphi|^2 + R_g\varphi^2\right] dv_g}{\left(\int_M \varphi^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}}.$$

The Yamabe invariant is a conformal invariant: if u is a smooth function, then

$$\mathrm{Y}(M,g) = \mathrm{Y}\left(M, e^{2u}g\right),$$

hence it only depends on the conformal class $[g] = \left\{e^{2u}g,\ u \in C^\infty(M)\right\}$ of the metric g.

When M is closed, the Yamabe invariant has the following geometric interpretation:

$$\mathrm{Y}(M,[g]) = \inf_{\tilde g\in[g]} \left\{ \frac{1}{\mathrm{vol}(M,\tilde g)^{1-\frac{2}{n}}} \int_M R_{\tilde g} dv_{\tilde g} \right\}.$$

According to the work of H. Yamabe, N. Trudinger, T. Aubin and R. Schoen, we can always find a metric $\tilde g \in [g]$ conformally equivalent to g such that

$$\frac{1}{\mathrm{vol}(M,\tilde g)^{1-\frac{2}{n}}} \int_M R_{\tilde g} dv_{\tilde g} = \mathrm{Y}(M,[g]).$$

The scalar curvature of such a metric $\tilde g$ is constant, and is equal to

$$R_{\tilde g} = \frac{\mathrm{Y}(M^n,[g])}{\mathrm{vol}(M^n,g)^{\frac{2}{n}}}.$$

We call such a metric a *Yamabe minimizer*. Using the Hölder inequality, we see that we always have

$$\mathrm{Y}(M^n,[g]) \leq \|R_g\|_{L^{\frac{n}{2}}},$$

with equality if and only if g is a Yamabe minimizer.

We can state two particular results of M. Gursky's articles [25] and [26] as follows:

THEOREM 1.1. – *Assume that (M^4, g) is a closed oriented manifold with positive Yamabe invariant.*

i) *If the traceless part of the Ricci curvature satisfies*

$$\int_M \left|\mathring{\mathrm{Ric}}_g\right|^2 dv_g \leq \frac{1}{12}\mathrm{Y}(M^4,[g])^2, \tag{1.2}$$

then

- *either its first Betti number $b_1(M^4)$ vanishes,*
- *or equality holds in (1.2), $b_1 = 1$, g is a Yamabe minimizer and (M^4, g) is conformally equivalent to a quotient of $\mathbb{S}^3 \times \mathbb{R}$.*

ii) *If the Weyl curvature satisfies*

$$\int_M |W_g|^2 \, dv_g \leq \frac{1}{24} \, \Upsilon(M^4, [g])^2, \tag{1.3}$$

then

- *either its second Betti number* $b_2(M^4)$ *vanishes,*
- *or equality holds in* (1.3)*,* $b_2 = 1$ *and* (M^4, g) *is conformally equivalent to* $\mathbb{P}^2(\mathbb{C})$ *endowed with the Fubini-Study metric.*

The norms of the curvature tensors are taken by considering them as symmetric operators on differential forms, for instance with the Einstein summation convention we have

$$|W|^2 = \frac{1}{4} W_{ijkl} W^{ijkl} \quad \text{and} \quad |\mathrm{Ric}|^2 = \mathrm{Ric}_{ij} \mathrm{Ric}^{ij}.$$

M. Gursky proved these two results by finding a good metric in the conformal class of g, for which some pointwise pinching holds. Then, by combining a Bochner-Weitzenböck equation with the pointwise pinching, he was able to prove the vanishing of harmonic forms.

The purpose of the article is to prove several generalizations of M. Gursky's result. Instead of trying to obtain a pointwise pinching, we will take advantage of the Sobolev inequality induced by the positivity of the Yamabe invariant. We first prove an integral version of the classical Bochner-Weitzenböck theorem (Theorem 2.2), which allows us to show that a large part of the Bochner theorem of S. Gallot and D. Meyer on the Betti numbers of manifolds with nonnegative curvature operator remains true if we only make the assumption in an integral sense:

THEOREM A. – *If* (M^n, g)*,* $n \geq 4$ *is a closed Riemannian manifold such that*

$$\|\rho_g\|_{L^{\frac{n}{2}}} \leq \frac{1}{n(n-1)} \, \Upsilon(M^n, [g]), \tag{1.4}$$

then for all $1 \leq k \leq \frac{n-3}{2}$ *or* $k = \frac{n}{2}$*,*

- *either its* k*th Betti number* $b_k(M^n)$ *vanishes,*
- *or equality holds in* (1.4) *and (up to a conformal change in the case* $k = \frac{n}{2}$*) the pointwise equality* $\rho_g = \frac{1}{n(n-1)} R_g$ *holds,* $1 \leq b_k \leq \binom{n}{k}$*, every harmonic* k*-form is parallel and* g *is a Yamabe minimizer.*

REMARK 1.2. – In Theorem A, as well as in the other theorems of the article, the two cases are not mutually exclusive, i.e., equality can hold in (1.4) while a number of Betti numbers vanish.

We also obtain an alternative proof of Theorem 1.1 based on our integral Bochner-Weitzenböck theorem, and several generalizations of M. Gursky's result to higher dimensions and higher degrees. In particular, we prove the following extension to higher dimensions of the first part of Theorem 1.1:

THEOREM B. – *If* (M^n, g)*,* $n \geq 5$*, is a compact Riemannian manifold with positive Yamabe invariant such that*

$$\left\|\mathring{\mathrm{Ric}}_g\right\|_{L^{\frac{n}{2}}} \leq \frac{1}{\sqrt{n(n-1)}} \, \Upsilon(M^n, [g]), \tag{1.5}$$