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Structured bundles define differential *K*-theory

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STRUCTURED BUNDLES DEFINE DIFFERENTIAL K-THEORY

by

James Simons & Dennis Sullivan

Abstract. — Complex bundles with connection up to isomorphism form a semigroup under Whitney sum which is far from being a group. We define a new equivalence relation (structured equivalence) so that stable isomorphism classes up to structured equivalence form a group which is describable in terms of the Chern character form plus some finite type invariants from algebraic topology. The elements in this group also satisfy two further somewhat contradictory properties: a locality or gluing property and an integrality property. There is interest in using these objects as prequantum fields in gauge theory and M-theory.

***Résumé* (Les fibrés structurés définissent la K-théorie différentielle).** — Les fibrés complexes à connexion forment, à isomorphisme près, un semi-groupe sous la somme de Whitney qui est loin d'être un groupe. Nous définissons une nouvelle relation d'équivalence (l'équivalence structurée) de manière à ce que les classes d'isomorphismes stable, à équivalence structurée près, forment un groupe qui puisse être décrit en termes de forme de caractère de Chern et de quelques invariants de type fini de la topologie algébrique. Les éléments de ce groupe satisfont également à deux propriétés en quelque sorte contradictoires : une propriété de localité ou de *gluing* et une propriété d'intégralité. Il semble intéressant d'utiliser ces objets en tant que champs pré-quantiques en théorie de gauge et en M-théorie.

Let M be the category whose objects are smooth manifolds and whose morphisms are smooth maps. We assume the manifolds are either compact manifolds possibly with boundary or diffeomorphic to those obtained from these by deleting some or all of the boundary components.

Let $K^\wedge$ denote the contravariant functor on M to abelian groups defined by equivalence classes of pairs (V, A) where V is a complex vector bundle and A is a connection on V. The equivalence relation is generated by stable isomorphisms of bundles with connection and by structured equivalence: namely any deformation of a connection A on a fixed bundle along any smooth path of connections so that the associated odd Chern Simons form is exact. Recall the exterior d of the Chern Simons form of a path

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of connections measures the change in the Chern Character form at the endpoints of the path.

We want to compute $K^\wedge$. Note an equivalence class in $K^\wedge$ has a precise total even form representing the Chern character plus some information related to a total odd cohomology class represented by Chern Simons forms which are closed but not exact.

Let $\mathrm{ch}(K)$ denote the contravariant functor on M to abelian groups defined by considering pairs $([V], C)$ where V represents an element of the K-theory of complex vector bundles and C is a total closed complex valued even dimensional form so that C represents the Chern character of V in rational cohomology.

We have the obvious map from $K^\wedge$ to $\mathrm{ch}(K)$ which assigns to the pair (V, A) of bundle with connection the pair $([V], C)$ where $[V]$ is the class of the stable vector bundle V in Atiyah's K-theory and C is the differential form defined by the Chern Weil curvature construction representing the total Chern character.

Let Torus be the functor on M to abelian complex Lie groups given by the odd cohomology with complex coefficients modulo the sublattice defined by considering all maps into G , the union over n of the n-dimensional complex linear groups, and by pulling back the desuspended Chern character class. This class is defined universally at level n defining G by desuspending the Chern character class of the bundle on suspension G defined using the identity map of G as a gluing function.

Theorem 1. — *The homomorphism from $K^\wedge$ to $\mathrm{ch}(K)$ is onto.*

The kernel of the homomorphism is the abelian complex Lie group Torus. We have the natural short exact sequence:

$$0 \to \text{Torus} \to K^\wedge \to \mathrm{ch}(K) \to 0. \tag{1}$$

Let k denote the kernel of the natural map from $K^\wedge$ to K, namely $(V, C) \to [V]$. Then from sequence (1), k maps with kernel Torus onto exact total even forms.

Let O = total odd forms modulo all closed forms in the cohomology classes of the sublattice above defining Torus. Then O maps via d with kernel Torus onto exact total even forms. The construction of $K^\wedge$ shows the kernel k is naturally isomorphic to O. In the detailed paper [1], O is denoted $\wedge^{odd}(X)/\wedge_G(X)$.

Theorem 2. — *There is the natural short exact sequence*

$$0 \to O \to K^\wedge \to K \to 0. \tag{2}$$

One may also show from the construction:

Theorem 3. — *$K^\wedge$ satisfies the Mayer-Vietoris property: if X is A union B with intersection C then given two elements a in $K^\wedge(A)$ and b in $K^\wedge(B)$ which restrict to the same element c in $K^\wedge(C)$, then there is an element x in $K^\wedge(X)$ which restricts to a and to b respectively.*

Consider E = all total even forms in the cohomology classes of the Chern characters of complex vector bundles. By Theorem 1 the map $K^\wedge \to E$ is surjective. Now consider k', the kernel of this map from $K^\wedge$ to E. Since $K^\wedge$ satisfies the Mayer-Vietoris property so does this kernel k'. One can show also that k' is a homotopy functor. Thus by Brown's representability theorem k' is represented by homotopy classes of maps into some space. Using this , the sequences above and side condition 1 in Remark 1 below leads to

Theorem 4. — *The kernel of the surjection of $K^\wedge$ onto E is naturally isomorphic to K-theory with coefficients in C/Z. Let us denote the latter by $K(C/Z)$. Then we have the natural short exact sequence:*

$$0 \to K(C/Z) \to K^\wedge \to E \to 0. \tag{3}$$

Now $K^\wedge$ is not a homotopy functor, but the change produced by an infinitesimal deformation v of a map can be computed. This change u is in $O =$ the kernel of $(K^\wedge \to K)$ because K is a homotopy functor. We know that du is the lie derivative of the Chern character form. So the following is natural and indeed true for $K^\wedge$:

Theorem 5. — *The change in $f^*(x)$ for x in $K^\wedge$ by an infinitesimal deformation v of a map f is obtained by contracting the Chern form of x by$v$ and projecting it to O inside $K^\wedge$.*

Remark 1. — *We have omitted two natural side conditions in the statements of Theorems 2 and 4 which should be noted.*

1. *The composition $K(C/Z) \to K^\wedge \to K$ using* (2) *and* (3) *is the Bockstein map in the Bockstein exact sequence for K-theory.*
2. *The composition $O \to K^\wedge \to E$ using* (2) *and* (3) *is exterior d.*

Conjecture. — *There is at most one functor $K^\wedge$ up to natural equivalence satisfying Theorems 1, 2, 3, 4 and 5 and the side conditions 1 and 2 in Remark 1.*

The presence of the homotopy property expressed by Theorem 5 in the conjecture above was inspired by conversations with Moritz Wiethaup. This homotopy property was not needed in our axioms characterizing ordinary differential cohomology [**2**]. The details of the proofs of the results here will appear soon [**1**]. We close by expressing on this occasion our appreciation of and admiration for the geometer Jean Pierre Bourguignon.

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Einstein metrics and magnetic monopoles

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EINSTEIN METRICS AND MAGNETIC MONOPOLES

by

Nigel Hitchin

For Jean Pierre Bourguignon on his 60th birthday

Abstract. — We investigate the geometry of the moduli space of centred magnetic monopoles on hyperbolic three-space, and derive using twistor methods some (incomplete) quaternionic Kähler metrics of positive scalar curvature. For the group SU(2) these have an orbifold compactification but we show that this is not the case for SU(3).

***Résumé* (Métriques d'Einstein et monopoles magnétiques).** — Nous étudions la géométrie des espaces de modules des monopoles maghétiques sur le 3-espace hyperbolique et nous en dérivons quelques métriques kähleriennes quaternioniques (incomplètes) de courbure scalaire positive, en utilisant des méthodes *twistor*. Celles-ci ont une compactification orbifolde pour le groupe SU(2) et nous montrons qu'il n'en est rien dans le cas du groupe SU(3).

1. Introduction

Over 20 years ago Jean Pierre Bourguignon and I were part of the team helping Arthur Besse to produce a state-of-the-art book on Einstein manifolds [**3**]. As might have been expected, the subject proved to be a moving target, and the contributors had to quickly assemble a number of appendices to cover material that came to light after all the initial planning. The last sentence of the final appendix refers to: "hyperkählerian metrics on finite dimensional moduli spaces", and so it seems appropriate to write here about some of the results which have followed on from that, and some questions that remain to be answered.

There is by now a range of gauge-theoretical moduli spaces which have natural hyperkähler metrics: the moduli space of instantons on $\mathbf{R}^4$ or the 4-torus or a K3

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surface [**16**], magnetic monopoles on $\mathbf{R}^3$ [**2**] and Higgs bundles on a Riemann surface [**12**]. The latter structure features prominently in the recent work of Kapustin and Witten on the Geometric Langlands correspondence [**15**]. Some of these metrics, in low dimensions, can be explicitly calculated, but even when this is not possible, the fact that these spaces are moduli spaces enables us to observe some geometrical properties which reflect their physical origin. In this paper we shall concentrate on the case of magnetic monopoles.

For monopoles in Euclidean space $\mathbf{R}^3$, there exist in certain cases explicit formulae (for example [**5**]), but in general we cannot write the metric down. Instead we can seek a geometrical means to describe the metrics; such a technique is provided by the use of twistor spaces, spectral curves and the symplectic geometry of the space of rational maps. This is documented in [**2**]. We review this in Section 2, drawing on new approaches to the symplectic structure.

We then shift attention to the hyperbolic version. The serious study of monopoles in hyperbolic space $\mathbf{H}^3$ was initiated long ago by Atiyah [**1**], who showed that there were many similarities with the Euclidean case. Yet the differential-geometric structure of the moduli space is still elusive, despite recent efforts [**18**], [**19**]. One would expect some type of quaternionic geometry which in the limit where the curvature of the hyperbolic space becomes zero approaches hyperkähler geometry. In Section 3 we give one approach to this, and show, following [**17**], how to resolve one of the problems that arises in attempting this – assigning a centre to a hyperbolic monopole.

The other problem, concerning a real structure on the putative twistor space, can currently be avoided only in the case of charge 2 and in Section 4 we produce, for the groups SU(2) and SU(3), quaternionic Kähler metrics on the moduli spaces of centred hyperbolic monopoles, generalizing the Euclidean cases computed in [**2**] and [**8**]. These metrics are expressed initially in twistor formalism, using the holomorphic contact geometry of certain spaces of rational maps, but we obtain some very explicit formulae as well.

For SU(2), these concrete self-dual Einstein metrics, originally introduced in [**14**], have nowadays found a new life in the area of 3-Sasakian geometry, Kähler-Einstein orbifolds and manifolds of positive sectional curvature. We consider briefly these aspects in the final section, and suggest where new examples might be found.

2. Euclidean monopoles

All of the hyperkähler moduli spaces mentioned above arise through the hyperkähler quotient construction. Recall that a hyperkähler metric on a manifold M^{4n} is defined by three closed 2-forms $\omega_1, \omega_2, \omega_3$ whose joint stabilizer at each point is conjugate to $Sp(n) \subset GL(4n, \mathbf{R})$. If a Lie group G acts on M, preserving the forms, then there usually exists a hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$. The quotient construction is the statement that the induced metric on $\mu^{-1}(0)/G$ is also hyperkähler.

For the moduli space of monopoles we use an infinite-dimensional version of this. The objects consist of connections A on a principal G-bundle over $\mathbf{R}^3$ together with a Higgs field ϕ, a section of the adjoint bundle. There are boundary conditions at infinity [**2**], in particular that $\|\phi\| \sim 1 - k/2r$, which imply that the connection on the sphere of radius R approaches a standard homogeneous connection as $R \to \infty$. The manifold M to which we apply the quotient construction then consists of pairs (A, ϕ) which differ from this standard connection by terms which decay appropriately, and in particular are in $\mathcal{L}^2$. This is formally an affine flat hyperkähler manifold where the closed forms ω_i are given by

$$\omega_i((\dot{A}_1, \dot{\phi}_1), (\dot{A}_2, \dot{\phi}_2)) = \int_{\mathbf{R}^3} dx_i \wedge \operatorname{tr}(\dot{A}_1 \dot{A}_2) + \int_{\mathbf{R}^3} *dx_i \wedge [\operatorname{tr}(\dot{\phi}_1 \dot{A}_2) - \operatorname{tr}(\dot{\phi}_2 \dot{A}_1)].$$

For the symplectic action of a group we take the group of gauge transformations which approach the identity at infinity suitably fast.

The zero set of the moment map in this case consists of solutions to the Bogomolny equations $F_A = *d_A\phi$, and the hyperkähler quotient is a bundle over the true moduli space of solutions – it is a principal bundle with group the automorphisms of the homogeneous connection at infinity. This formal framework has to be supported by analytical results of Taubes to make it work rigorously.

When $G = \mathrm{SU}(2)$, the connection on a large sphere has structure group $U(1)$ and Chern class k, which is called the monopole charge. The hyperkähler quotient is a manifold of dimension $4k$ which is a circle bundle over the true moduli space. It has a complete metric which is invariant under the Euclidean group and the circle action (completeness comes from the Uhlenbeck weak compactness theorem, one use of gauge theoretical results to shed light on metric properties). The gauge circle action in fact preserves the hyperkähler forms $\omega_1, \omega_2, \omega_3$, and its moment map defines a centre in $\mathbf{R}^3$. The $(4k-4)$-dimensional hyperkähler quotient can then be thought of as the moduli space of centred monopoles.

For charge 2, by using a variety of techniques [**2**], one can determine the metric explicitly. It has an action of SO(3) and may be written as

$$g = (abc)^2 d\eta^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{1}$$

where

$$ab = -2k(k')^2 K\frac{dK}{dk} \quad bc = ab - 2(k'K)^2 \quad ca = ab - 2(k'K)^2$$

$$\eta = -K'/\pi K \qquad K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

and $\sigma_1, \sigma_2, \sigma_3$ are the standard left-invariant forms on SO(3).

Differentiably, this manifold can be understood in terms of the unit sphere in the irreducible 5-dimensional representation space of SO(3). For each axis there is, up to a scalar multiple, a unique axially symmetric vector in this representation and these trace out two copies of $\mathbf{RP}^2 \subset S^4$. The centred moduli space is the complement of one of these, the removed point being the axis joining two widely separated monopoles. The other $\mathbf{RP}^2$ parametrizes axially symmetric monopoles, which are (for any value of charge) uniquely determined by their axis.

For the group $G = \mathrm{SU}(3)$ we consider a Higgs field which asymptotically has two equal eigenvalues. On a large sphere the eigenspace is a rank two bundle with first Chern class k, again called the charge. When $k = 2$, Dancer computed this metric [**7**]. For centred monopoles it is eight-dimensional with an $\mathrm{SO}(3)\times\mathrm{PSU}(2)$ action, the first factor from the geometric action of rotations and the second from the automorphisms of the connection at infinity. Explicitly it can be written as follows:

$$g = \frac{1}{4}\sum_i (x(1+px)m_im_i + y(1+py)n_in_i + 2pxym_in_i)$$

where

$$\begin{aligned}
m_1 &= -f_1df_1 + f_2df_2 \quad m_2 = (f_1^2 - f_2^2)\sigma_3\\
m_3 &= \frac{1}{px}[(pyf_3^2 - (1+py)f_1^2)\sigma_2 + f_3f_1\Sigma_2]\\
m_4 &= -\frac{1}{1+px+py}[(pyf_3^2 - (1+py)f_2^2)\sigma_1 + f_2f_3\Sigma_1]\\
n_1 &= \frac{1}{py}(-pxf_2df_2 + (1+px)f_1df_1)\\
n_2 &= \frac{1}{py}[((1+px)f_1^2 - pxf_2^2)\sigma_3 - f_1f_2\Sigma_3]\\
n_3 &= (f_1^2 - f_3^2)\sigma_2\\
n_4 &= \frac{1}{1+px+py}[(pxf_2^2 - (1+px)f_3^2)\sigma_1 + f_2f_3\Sigma_1]
\end{aligned}$$

with σ_i, Σ_i invariant one-forms on $\mathrm{SO}(3)\times\mathrm{SU}(2)$, and

$$f_1 = -\frac{D\,\mathrm{cn}(3D,k)}{\mathrm{sn}(3D,k)} \quad f_2 = -\frac{D\,\mathrm{dn}(3D,k)}{\mathrm{sn}(3D,k)} \quad f_3 = -\frac{D}{\mathrm{sn}(3D,k)},$$

$$x = \frac{1}{D^3}\int_0^{3D}\frac{\mathrm{sn}^2(u)}{\mathrm{dn}^2(u)}du \quad y = \frac{1}{D^3}\int_0^{3D}\mathrm{sn}^2(u)du.$$

and $p = f_1f_2f_3$ for $D < 2K/3$.

Clearly there are limits to extracting information from formulae like these. Nevertheless, the restriction to certain submanifolds can be useful.

2.1. Twistor spaces. — Penrose's twistor theory provides a method for transforming the equations of a hyperkähler metric into holomorphic geometry. The idea is that the three closed two-forms of a hyperkähler manifold M can be arranged as $\omega_1, \omega_2 + i\omega_3$ which define a complex structure I for which $\omega_2 + i\omega_3$ is a holomorphic symplectic two-form and ω_1 a Kähler form. The other choices give complex structures J, K; more generally for a point $\mathbf{x} \in S^2$, $(x_1 I + x_2 J + x_3 K)^2 = -1$ and defines a complex structure.

The twistor space is the product $Z = M \times S^2$. It has a complex structure $((x_1 I + x_2 J + x_3 K), \mathbf{I})$ where $\mathbf{I}$ is the complex structure on $S^2 = \mathbf{C}\mathrm{P}^1$. The projection $p : Z \to \mathbf{C}\mathrm{P}^1$ to the second factor is holomorphic, and the fibre is M with the structure of a holomorphic symplectic manifold. There is a real structure $(m, (x_1, x_2, x_3)) \to (m, -(x_1, x_2, x_3))$. To recover the space M one sees that for $m \in M$, (m, S^2) is a holomorphic section of the projection p and M is then a component of the space of real sections.

We shall describe here how to construct the twistor space for the moduli space of SU(2) monopoles on $\mathbf{R}^3$ (see [**2**]). This involves the link with rational maps. Consider a straight line $\mathbf{x} = \mathbf{a} + t\mathbf{u}$ and the ordinary differential equation along the line $(\nabla_{\mathbf{u}} - i\phi)s = 0$. Since asymptotically $\phi \sim \mathrm{diag}(i, -i)$, there is a solution s_0 which decays exponentially at $+\infty$. Choose another solution s_1 with $\langle s_0, s_1 \rangle = 1$ using the SU(2)-invariant skew form. This is well-defined modulo the addition of a multiple of s_0. Now take s_0', a solution which decays at $-\infty$, then $s_0' = as_0 + bs_1$. There are normalizations at infinity which make the coefficient b well-defined.

Now take all lines in a fixed direction $(1, 0, 0)$. We split $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ with coordinates $(z, t) = (x_1 + ix_2, x_3)$, and then write $s_0'(z, t) = a(z)s_0(z, t) + b(z)s_1(z, t)$. The Bogomolny equations imply that the coefficients are holomorphic in z, and furthermore the boundary conditions tell us that for a charge k monopole $b(z)$ is a polynomial of degree k. Take $p(z)$ to be the unique polynomial of degree $k-1$ such that $p(z) = a(z)$ modulo $b(z)$ and define

$$S(z) = \frac{p(z)}{b(z)} = \frac{a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}}{b_0 + b_1 z + \ldots \cdots + b_{k-1} z^{k-1} + z^k}.$$

It is a theorem that this gives a diffeomorphism between the moduli space of monopoles and the space R_k of rational maps $S : \mathbf{C}\mathrm{P}^1 \to \mathbf{C}\mathrm{P}^1$ of degree k which take ∞ to 0. Note that the denominator vanishes when $s_0' = as_0$ – when a solution exists which decays at both ends of the line. Such lines are called spectral lines.

We can carry out the above isomorphism for lines in any direction in $\mathbf{R}^3$ which yields a 2-sphere of complex symplectic structures. The set of spectral lines then forms a k-fold cover of S^2 which is called the spectral curve. It is more than just an