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CENTRAL LIMIT THEOREMS FOR THE BROWNIAN MOTION ON LARGE UNITARY GROUPS

by Florent Benaych-Georges

Abstract. — In this paper, we are concerned with the large n limit of the distributions of linear combinations of the entries of a Brownian motion on the group of $n \times n$ unitary matrices. We prove that the process of such a linear combination converges to a Gaussian one. Various scales of time and various initial distributions are considered, giving rise to various limit processes, related to the geometric construction of the unitary Brownian motion. As an application, we propose a very short proof of the asymptotic Gaussian feature of the entries of Haar distributed random unitary matrices, a result already proved by Diaconis *et al.*

Résumé (*Théorèmes centraux limite pour le mouvement brownien sur le groupe unitaire de grande taille*)

Dans cet article, on considère la loi limite, lorsque n tend vers l'infini, de combinaisons linéaires des coefficients d'un mouvement Brownien sur le groupe des matrices unitaires $n \times n$. On prouve que le processus d'une telle combinaison linéaire converge vers un processus gaussien. Différentes échelles de temps et différentes lois initiales sont considérées, donnant lieu à plusieurs processus limites, liés à la construction géométrique du mouvement Brownien unitaire. En application, on propose une preuve très courte du caractère asymptotiquement gaussien des coefficients d'une matrice unitaire distribuée selon la mesure de Haar, un résultat déjà prouvé par Diaconis *et al.*

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Introduction

There is a natural definition of Brownian motion on any compact Lie group, whose distribution is sometimes called the heat kernel measure. Mainly due to its relations with the object from free probability theory called the free unitary Brownian motion and with the two-dimentional Yang-Mills theory, the Brownian motion on large unitary groups has appeared in several papers during the last decade. Rains, in [25], Xu, in [29], Biane, in [3, 4] and Lévy and Maïda, in [18, 19], are all concerned with the asymptotics of the spectral distribution of large random matrices distributed according to the heat kernel measure. Also, in [12], Demni makes use of the unitary Brownian motion in the study of Jacobi processes, and, in [2], Lévy and the author of the present paper construct a continuum of convolutions between the classical and free ones based on the conjugation of random matrices with a unitary Brownian motion. In this paper, we are concerned with the asymptotic distributions of linear combinations of the entries of an $n \times n$ unitary Brownian motion as n tends to infinity.

We first give the joint limit distribution, as n tends to infinity, of (possibly rescaled) random processes of the type $(\mathrm{Tr}[A(V_t - I)])_{t\geq 0}$ for $(e^{-t/2}V_t)_{t\geq 0}$ a Brownian motion staring at I on the group of unitary $n \times n$ matrices and A an $n \times n$ matrix (Theorem 1.2). This theorem is the key result of the paper, since specifying the choice of the matrices A and randomizing them will then allow us to prove all other results. As a first example, it allows us to find out, for any sequence $(\alpha_n)_n$ of positive numbers with a limit $\alpha \in [0, +\infty]$, the limit distribution of any upper-left corner of $\sqrt{n/\alpha_n}(V_{\log(\alpha_n t+1)} - I)_{t\geq 0}$ (Corollary 1.4): for small scales of time (i.e. when $\alpha = 0$), the limit process is purely skew-Hermitian, whereas for large scales of time ($\alpha = +\infty$), the limit process is a standard complex matricial Brownian motion. For intermediate scales of time ($0 < \alpha < +\infty$), the limit process is an interpolation between these extreme cases. The existence of these three asymptotic regimes can be explained by the fact that the unitary Brownian motion is the "wrapping", on the unitary group, of a Brownian motion on the tangent space of this group at I (which is the space of skew-Hermitian matrices), and that as the time goes to infinity, its distribution tends to the Haar measure (for which the upper-left corners are asymptotically distributed as standard complex Gaussian random matrices).

Secondly, we consider a unitary Brownian motion $(e^{-t/2}V_t)_{t\geq 0}$ whose initial distribution is the uniform measure on the group of permutation matrices: its rows are exchangeable, as its columns. In this case, for any positive sequence (α_n) and any positive integer p, the $p \times p$ upper left corner of $(\sqrt{n/\alpha_n} V_{\log(\alpha_n t+1)})_{t\geq 0}$ converges to a standard complex matricial Brownian motion (Corollary 1.9).

Since the unitary Brownian motion distributed according to the Haar measure at time zero has a stationary distribution, our results allow us to give very short proofs of some well-known results of Diaconis *et al*, first proved in [10], about the asymptotic normality of linear combinations of the entries of uniform random unitary matrices (Theorem 1.11 and Corollary 1.12).

It is clear that the same analysis would give similar results for the Brownian motion on the orthogonal group. For notational brevity, we chose to focus on the unitary group.

Let us now present briefly what problems underlie the asymptotics of linear combinations of the entries of a unitary Brownian motion.

Asymptotic normality of random unit vectors and unitary matrices: The historical first result in this direction is due to Émile Borel, who proved a century ago, in [5], that, for a uniformly distributed point $(X_1, \dots, X_n)$ on the unit euclidian sphere $\mathbb{S}^{n-1}$, the scaled first coordinate $\sqrt{n}X_1$ converges weakly to the standard Gaussian distribution as the dimension n tends to infinity. As explained in the introduction of the paper [10] of Diaconis *et al.*, this says that the features of the "microcanonical" ensemble in a certain model for statistical mecanics (uniform measure on the sphere) are captured by the "canonical" ensemble (Gaussian measure). Since then, a long list of further-reaching results about the entries of uniformly distributed random orthogonal or unitary matrices have been obtained. The most recent ones are the previously cited paper of Diaconis *et al.*, the papers of Meckes and Chatterjee [20, 6], the paper of Collins and Stolz [9] and the paper of Jiang [17], where the point of view is slightly different. In the present paper, we give a new, quite short, proof of the asymptotic normality of the linear combinations of the entries of uniformly distributed random unitary matrices, but we also extend these investigations to the case where the distribution of the matrices is not the Haar measure but the heat kernel measure, with any initial distribution and any rescaling of the time.

Second order freeness: A theory has been developed these last five years about Gaussian fluctuations (called *second order limits*) of traces of large random matrices around their limits, the most emblematic articles in this theory being [21, 23, 22, 8]. The results of this paper can be related to this theory, even though, technically speaking, we do not consider the powers of the matrices here[(1)].

Brownian motion on the Lie algebra and Itō map: The unitary Brownian motion is a continuous random process taking values on the unitary group,

(1) The reason is that the constant matrices we consider here, like $\sqrt{n}\times$(an elementary $n \times n$ matrix), have no bounded moments of order higher than two: our results are the best ones that one could obtain with such matrices.

which has independent and stationary multiplicative increments. The most constructive way to define it is to consider a standard Brownian motion $(B_t)_{t\geq 0}$ on the tangent space of the unitary group at the identity matrix and to take its image by the *Itō map* (whose inverse is sometimes called the *Cartan map*), i.e. to wrap[(2)] it around the unitary group: the process $(U_t)_{t\geq 0}$ obtained is a unitary Brownian motion starting at I. Our results give us an idea of the way the Itō map alterates the process $(B_t)_{t\geq 0}$ at different scales of time. Moreover, the question of the choice of a rescaling of the time (depending on the dimension) raises interesting questions (see Remark 1.1).

Notation. — For each $n \geq 1$, $\mathbb{U}_n$ shall denote the group of $n \times n$ unitary matrices. The identity matrix will always be denoted by I. For each complex matrix M, M^* will denote the adjoint of M. We shall call a standard complex Brownian motion a complex-valued process whose real and imaginary parts are independent standard real Brownian motions divided by $\sqrt{2}$. For all $k \geq 1$, the space of continuous functions from $[0, +\infty)$ to $\mathbb{C}^k$ will be denoted by $\mathscr{C}([0, +\infty), \mathbb{C}^k)$ and will be endowed with the topology of the uniform convergence on every compact interval.

1. Statement of the results

1.1. Brief presentation of the Brownian motion on the unitary group. — There are several ways to construct the Brownian motion on the unitary group[(3)]. For the one we choose here, all facts can easily be recovered by the use of the matricial Itō calculus, as exposed in Section 2.1.

Let n be a positive integer and ν_0 a probability measure on the group of unitary $n \times n$ matrices. We shall call a *unitary Brownian motion with initial law* ν_0 any random process $(U_t)_{t\geq 0}$ with values on the space of $n \times n$ complex matrices such that U_0 is ν_0-distributed and $(U_t)_{t\geq 0}$ is a strong solution of the stochastic differential equation

$$\mathrm{d}U_t = i\mathrm{d}H_t U_t - \frac{1}{2} U_t \mathrm{d}t, \tag{1}$$

(2) For G a matricial Lie group with tangent space $\mathfrak{g}$ at I, the "wrapping" w_γ, on G, of a continuous and piecewise smooth path $\gamma : [0, +\infty) \to \mathfrak{g}$ such that $\gamma(0) = 0$ is defined by $w_\gamma(0) = I$ and $w'_\gamma(t) = w_\gamma(t) \times \gamma'(t)$. It B is a Brownian motion on $\mathfrak{g}$ and $(B_n)_{n\geq 1}$ is a sequence of continuous, piecewise affine interpolations of B with a step tending to zero as n tends to infinity, then the sequence w_{B_n} converges in probability to a process which doesn't depend on the choice of the interpolations and which is a Brownian motion on G [16, Sect. VI.7], [26, Eq. (35.6)], [14].

(3) See [15, 28, 16, 26]. A very concise and elementary definition is also given in [25].