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EXTENSION OF ESTERMANN'S THEOREM TO EULER PRODUCTS ASSOCIATED TO A MULTIVARIATE POLYNOMIAL

by Ludovic Delabarre

Abstract. — Given a multivariate polynomial $h(X_1, \ldots, X_n)$ with integral coefficients verifying an hypothesis of analytic regularity (and satisfying $h(\mathbf{0}) = 1$), we determine the maximal domain of meromorphy of the Euler product $\prod_{p \text{ prime}} h\left(p^{-s_1}, \ldots, p^{-s_n}\right)$ and the natural boundary is precisely described when it exists. In this way we extend a well known result for one variable polynomials due to Estermann from 1928. As an application, we calculate the natural boundary of the multivariate Euler products associated to a family of toric varieties.

Résumé (*Extension du théorème d'Estermann aux produits eulériens associés à un polynôme de plusieurs variables*)

Etant donné un polynôme de plusieurs variables $h(X_1, \ldots, X_n)$ à coefficients entiers vérifiant une hypothèse de régularité analytique (et vérifiant $h(\mathbf{0}) = 1$), on détermine le domaine maximal de méromorphie du produit eulérien $\prod_{p \text{ premier}} h\left(p^{-s_1}, \ldots, p^{-s_n}\right)$ et la frontière naturelle de méromorphie est décrite précisément lorsqu'elle existe. De

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cette façon on généralise un résultat célèbre de 1928 concernant les polynômes d'une variable due à Estermann. En guise d'application, on détermine la frontière naturelle de produits eulériens de plusieurs variables associés à une famille de variétés toriques.

1. Introduction

A classic result of Estermann ([7]) from 1928 characterized precisely when an Euler product $Z(s)=\prod_p h(p^{-s})$, determined by a polynomial $h(X)\in\mathbb{Z}[X]$ with $h(0)=1$, admits a meromorphic extension to $\mathbb{C}$. In addition, Estermann showed that if this property is not satisfied, then the Euler product has a natural boundary as a meromorphic function which he identified exactly.

This article extends Estermann's theorem to all Euler products $Z(\mathbf{s})=h(p^{-s_1},\ldots,p^{-s_n})$ determined by any polynomial $h(X_1,\ldots,X_n)\in\mathbb{Z}[X_1,\ldots,X_n]$ verifying an hypothesis of analytic regularity which is mostly satisfied (see Definition 3) when $n\geq 2$ and $h(\mathbf{0})=1$. Thus, we characterize precisely the natural boundary of $Z(\mathbf{s})$ when h is not cyclotomic[(1)] (see Definition 1), that is, the boundary of a maximal domain on which it can be meromorphically continued.

1.1. Notations. — For two positive integers r and n we define:

$$h\left(X_1,\ldots,X_n\right):=1+a_1X_1^{\alpha_{11}}X_2^{\alpha_{21}}\cdots X_n^{\alpha_{n1}}+\ldots+a_rX_1^{\alpha_{1r}}X_2^{\alpha_{2r}}\cdots X_n^{\alpha_{nr}};$$

$$Z\left(\mathbf{s}\right):=\prod_p h\left(p^{-s_1},p^{-s_2},\ldots,p^{-s_n}\right)\ \text{ for } \mathbf{s}=\left(s_1,\ldots,s_n\right)\in\mathbb{C}^n,$$

where a_j for $j=1,\ldots,r$ are integers and $\alpha_{\ell j}$ for $j=1,2,\ldots,r$ and $\ell=1,\ldots,n$ are non negative integers.

We also fix the following notations throughout the article.

We put $\boldsymbol{\alpha}:=(\alpha_{\ell j})_{(\ell,j)\in\{1,\ldots,n\}\times\{1,\ldots,r\}}\in\mathbb{M}_{n,r}(\mathbb{N})$ the matrix encoding the exponents of h whose rows for $\ell\in\{1,\ldots,n\}$ are written $\boldsymbol{\alpha}_{\ell\cdot}:=(\alpha_{\ell 1},\ldots,\alpha_{\ell r})$ and columns for $j\in\{1,\ldots,r\}$ are written $\boldsymbol{\alpha}_{\cdot j}:={}^t(\alpha_{1j},\ldots,\alpha_{nj})$.

For $j\in\{1,\ldots,r\}$ we set $\mathbf{X}^{\boldsymbol{\alpha}_{\cdot j}}:=X_1^{\alpha_{1j}}X_2^{\alpha_{2j}}\cdots X_n^{\alpha_{nj}}$ so that $h(\mathbf{X})=1+\sum_{j=1}^r a_j\mathbf{X}^{\boldsymbol{\alpha}_{\cdot j}}$.

For all $\mathbf{y}\in\mathbb{R}^r$ we will write $\|\mathbf{y}\|:=\displaystyle\sum_{j=1}^r y_j$ and $\langle\mathbf{y}\rangle=\mathbb{R}\mathbf{y}$.

For $\mathbf{s}=(s_1,\cdots,s_n)\in\mathbb{C}^n$, and $\ell\in\{1,\cdots,n\}$, we put:

$$\sigma_\ell:=\Re\left(s_\ell\right);\ \tau_\ell:=\Im\left(s_\ell\right);\ \boldsymbol{\sigma}:=\Re\left(\mathbf{s}\right):=\left(\sigma_1,\cdots,\sigma_n\right);\ \boldsymbol{\tau}:=\Im\left(\mathbf{s}\right):=\left(\tau_1,\ldots,\tau_n\right).$$

(1) If h is cyclotomic one can check that $Z(\mathbf{s})=h(p^{-s_1},\ldots,p^{-s_n})$ continues meromorphically to $\mathbb{C}^n$ (see Remark 2).

We write for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$ and $\boldsymbol{w} = {}^t(w_1, \ldots, w_m)$ the classical matrix product between $\boldsymbol{\nu}$ and $\boldsymbol{w}$:

$$\boldsymbol{\nu} \cdot \boldsymbol{w} := \sum_{i=1}^{m} \nu_i w_i.$$

For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$ and $\mathbf{s} = (s_1, \ldots, s_n)$, we will use throughout the paper the following equality

$$\sum_{j=1}^{r} \beta_j \left(\mathbf{s} \cdot \boldsymbol{\alpha}_{j\cdot}\right) = \sum_{\ell=1}^{n} s_\ell \left(\boldsymbol{\alpha}_{\ell\cdot} \cdot {}^t\boldsymbol{\beta}\right) := \mathbf{s} \cdot \boldsymbol{\alpha} \cdot {}^t\boldsymbol{\beta};$$

which results from the classical identity $(\mathbf{s} \cdot \boldsymbol{\alpha}) \cdot {}^t\boldsymbol{\beta} = \mathbf{s} \cdot (\boldsymbol{\alpha} \cdot {}^t\boldsymbol{\beta})$.

1.2. Statement of main results. — We first recall the classic result of Estermann [7] for one variable polynomials.

Theorem. (Estermann) *Let* $h(X) = 1 + \sum_{m=1}^{r} b_m X^m = \prod_{m=1}^{r} (1 - \alpha_m X) \in \mathbb{Z}[X]$. *Let* $f(s) = \prod_p h(p^{-s})$, *which converges for* $\Re(s) > 1$. *Then:*

(i) $f(s)$ *can be meromorphically extended to* $\Re(s) > 0$.

(ii) *If* $|\alpha_m| = 1$ *for all* $m = 1, .., r$, *then* $f(s)$ *can be extended to* $\mathbb{C}$. *Otherwise,* $\Re(s) = 0$ *is a natural boundary for* f *(i.e. for each point* $s = it$ *on this vertical line,* f *cannot be extended as a meromorphic function on any neighborhood of* $s = it$*).*

Remark 1. — G. Dahlquist [5] generalized this result to analytic functions h with isolated singularities within the unit circle. Later, deep work by Kurokawa [8], [9] and Moroz [12] extended Estermann's result by allowing polynomials $h(X)$ whose coefficients were integral linear combinations of complex numbers associated to characters of finite dimensional representations of a topological group.

Estermann's main result leads naturally to the following basic definition.

Definition 1. — In this paper a polynomial $Q(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$ is said to be cyclotomic if there exists $m \in \mathbb{N}^*$, and for each $k \in \{1, \ldots, m\}$, some $\boldsymbol{\gamma}_k \in \mathbb{N}^n$ and $j_k \in \mathbb{N}$ such that we have

$$Q(X_1, \ldots, X_n) = \prod_{k=1}^{m} \Phi_{i_k}(\mathbf{X}^{\boldsymbol{\gamma}_k})^{j_k}; \tag{1}$$

where, for $i_k \in \mathbb{N}^*$, $\Phi_{i_k}(X)$ denotes the classical i_k-th cyclotomic polynomial in the usual sense.

REMARK 2. — If $Q(X_1,\ldots,X_n) \in \mathbb{Z}[X_1,\ldots,X_n]$ is cyclotomic, then there exists a finite subset I of $\mathbb{N}^n \setminus \{\mathbf{0}\}$ such that

$$Q(X_1,\ldots,X_n) = \prod_{\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)\in I} \left(1-\mathbf{X}^{\boldsymbol{\lambda}}\right)^{\gamma(\boldsymbol{\lambda})}$$

with $\gamma(\boldsymbol{\lambda}) \in \mathbb{Z}$ for each $\boldsymbol{\lambda} \in I$. Thus we have for $\sigma_\ell > 1$ ($\ell \in \{1,\ldots,n\}$):

$$\prod_p Q\left(p^{-s_1},\ldots,p^{-s_n}\right) = \prod_{\boldsymbol{\lambda}\in I} \zeta\left(\mathbf{s}\cdot {}^t\boldsymbol{\lambda}\right)^{-\gamma(\boldsymbol{\lambda})},$$

where $\zeta(s)$ denotes the Riemann zeta function. As a result, it is clear that this Euler product meromorphically extends to $\mathbb{C}^n$ as a finite product of classical Riemann zeta functions.

Moreover, if two polynomials $h_1(X_1,\ldots,X_n)$ and $h_2(X_1,\ldots,X_n)$ are such that:

$$h_1(X_1,\ldots,X_n) = h_2(X_1,\ldots,X_n)\,Q(X_1,\ldots,X_n)$$

for some cyclotomic polynomial Q, then the maximal domains of meromorphic continuation of the Euler products $\prod_p h_1\left(p^{-s_1},p^{-s_2},\ldots,p^{-s_n}\right)$ and $\prod_p h_2\left(p^{-s_1},p^{-s_2},\ldots,p^{-s_n}\right)$ coincide.

So from now on, it suffices to assume the following:

The polynomial h is not constant and has no cyclotomic factors.

DEFINITION 2. — Suppose that the polynomial h is not constant and has no cyclotomic factors.

For all $\delta \geq 0$ we put $\mathbf{W}(\delta) = \{\mathbf{s}\in\mathbb{C}^n : \boldsymbol{\sigma}\cdot\boldsymbol{\alpha}_{.j} > \delta, \forall j \in \{1,\cdots,r\}\}$.

For a polynomial h in $n \geq 1$ variable(s), we first observe that $Z(\mathbf{s})$ defines a holomorphic function of $\mathbf{s}$ in the domain $\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}_{.j} > 1, (j = 1,\ldots,r)$. In [2], G. Bhowmik, D. Essouabri and B. Lichtin showed that there is a meromorphic continuation of $Z(\mathbf{s})$ to $\mathbf{W}(0)$. They did so by proving the following result.

Theorem (Bhowmik-Essouabri-Lichtin**)** *For each $\delta > 0$, there exists a bounded Euler product $G_\delta(\mathbf{s})$, absolutely convergent on $\mathbf{W}(\delta)$ such that:*

$$(2)\quad Z(\mathbf{s}) = \prod_{\substack{\boldsymbol{\beta}=(\beta_1,\cdots,\beta_r)\in\mathbb{N}^r\\ 1\leq\|\boldsymbol{\beta}\|\leq[\delta^{-1}]}} \zeta\left(\mathbf{s}\cdot\boldsymbol{\alpha}\cdot{}^t\boldsymbol{\beta}\right)^{\gamma(\boldsymbol{\beta})} G_\delta(\mathbf{s});\ \textit{where } \{\gamma(\boldsymbol{\beta}) : \boldsymbol{\beta}\in\mathbb{N}^r\} \subset \mathbb{Z}.$$

In fact, their result is somewhat stronger. They also showed that the function $Z(\mathbf{s})$ does not admit a meromorphic continuation to $\mathbf{W}(\delta)$ for any $\delta < 0$. This followed from the fact that $\mathbf{0}$ is an accumulation point of zeros or poles of the one variable function $t \longmapsto Z(t\cdot\boldsymbol{\theta})$ for almost all direction $\boldsymbol{\theta}\in\mathbb{R}^n$.

Before announcing the main result, we will first introduce a definition.