

Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

GLOBAL INFINITE ENERGY SOLUTIONS FOR THE CUBIC WAVE EQUATION

Nicolas Burq & Laurent Thomann & Nikolay Tzvetkov

**Tome 143
Fascicule 2**

2015

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique
pages 301-313

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 2, tome 143, juin 2015

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Maison de la SMF Case 916 - Luminy 13288 Marseille Cedex 9 France smf@smf.univ-mrs.fr	Hindustan Book Agency O-131, The Shopping Mall Arjun Marg, DLF Phase 1 Gurgaon 122002, Haryana Inde	AMS P.O. Box 6248 Providence RI 02940 USA www.ams.org
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Tarifs

Vente au numéro : 43 € (\$ 64)
Abonnement Europe : 176 €, hors Europe : 193 € (\$ 290)
Des conditions spéciales sont accordées aux membres de la SMF.

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Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • <http://smf.emath.fr/>

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ISSN 0037-9484

Directeur de la publication : Marc PEIGNÉ

GLOBAL INFINITE ENERGY SOLUTIONS FOR THE CUBIC WAVE EQUATION

BY NICOLAS BURQ, LAURENT THOMANN & NIKOLAY TZVETKOV

ABSTRACT. — We prove the existence of infinite energy global solutions of the cubic wave equation in dimension greater than 3. The data is a typical element on the support of suitable probability measures.

RÉSUMÉ (*Solutions globales d'énergie infinie pour l'équation des ondes cubique*)

On considère l'équation des ondes cubique sur un tore de dimension supérieure à 3, et on montre l'existence de solutions globales d'énergie infinie. La condition initiale de l'équation est un élément typique du support d'une mesure de probabilité.

Texte reçu le 30 octobre 2012, accepté le 22 avril 2013.

NICOLAS BURQ, Laboratoire de Mathématiques, UMR 8628 du CNRS, Bât. 425, Université Paris Sud, 91405 Orsay Cedex, France and Ecole Normale Supérieure, 45, rue d'Ulm, 75005 Paris, Cedex 05, France, UMR 8553 du CNRS •

E-mail : nicolas.burq@math.u-psud.fr

LAURENT THOMANN, Laboratoire de Mathématiques J. Leray, UMR 6629 du CNRS, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France •

E-mail : laurent.thomann@univ-nantes.fr

NIKOLAY TZVETKOV, University of Cergy-Pontoise, UMR CNRS 8088, Cergy-Pontoise, F-95000 • *E-mail :* nikolay.tzvetkov@u-cergy.fr

2010 Mathematics Subject Classification. — 35BXX, 37K05, 37L50.

Key words and phrases. — Nonlinear wave equation, random data, weak solutions, global solutions.

L.T. was partly supported by the grant ANR-10-JCJC 0109 and N.T. by the ERC grant Dispeq.

1. Introduction

This paper is a higher dimensional sequel of the recent article [10] by the first and the third authors (and also of [8, 9, 5]). As such it aims to construct global in time solutions of the cubic wave equation with low regularity (infinite energy) random initial data. To the best of our knowledge such a regularity is out of reach of the present deterministic methods. The major difference between the present paper and [10] is that here we only establish existence results and in particular no uniqueness statement is proven. Let us recall that in [10] a suitable uniqueness and a probabilistic continuity of the flow were proven. This result was followed by more recent results by Nahmod-Pavlovic-Staffilani [15] on the 2 and 3-dimensional homogeneous Navier-Stokes equation, where the authors obtain strong (in 2-d) and weak (in 3-d) results, and in turn, here we are inspired by this latter 3-d weak-existence result. Related weak-existence results had been already used in the context of the randomly forced Navier-Stokes equation by Da Prato-Debussche [12] and the Euler equation by Albeverio-Cruzeiro [1], using more sophisticated probabilistic tools (Prokhorov and Skorohod Theorems). This approach may be seen as the analogue in the random setting of the Leray compactness method for constructing solutions of nonlinear evolution equations. It has the advantage to require less regularity on the initial data, one allows infinite energy while the Leray method requires finite energy of the data. It should however be emphasised that as in the Leray method our approach still makes a crucial use of the energy functional. In this paper we will only need an invariance property for the linear evolution combined with large deviation estimates on the nonlinear part which are much easier to achieve than the invariance properties as in [12, 1]. Let us now describe our model. Let $d \geq 4$ and consider the cubic wave equation on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$

$$(1.1) \quad \begin{cases} \partial_t^2 u - \Delta u + u^3 = 0, & (t, x) \in \mathbb{R} \times \mathbb{T}^d, \\ (u, \partial_t u)(0, \cdot) = (u_0, u_1) \in \mathcal{H}^s, \end{cases}$$

where $\Delta := \Delta_{\mathbb{T}^d}$ is the Laplace operator and

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{T}^d) := H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d).$$

Denote by $s_c = (d-2)/2$ the critical (scaling) Sobolev index for (1.1). Then one can show that (1.1) is well-posed in \mathcal{H}^s for $s > s_c$ ([13]) and ill-posed when $s < s_c$ ([13, 11, 14]). See the introduction of [10] for more details. The energy of (1.1) reads

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{T}^d} (|\nabla u|^2 + (\partial_t u)^2) + \frac{1}{4} \int_{\mathbb{T}^d} u^4,$$

thus with deterministic compactness methods due to Leray (see *e.g.* Lebeau [14, Section 6] for the application of the method in the context of (1.1)), we

can construct global weak solutions to (1.1) so that

$$(u, \partial_t u) \in \mathcal{C}_w(\mathbb{R}; H^1(\mathbb{T}^d) \cap L^4(\mathbb{T}^d)) \times \mathcal{C}_w(\mathbb{R}; L^2(\mathbb{T}^d)),$$

(here \mathcal{C}_w means weak continuity in time) and $\mathcal{E}(u)(t) \leq \mathcal{E}(u)(0)$ for all $t \in \mathbb{R}$. Observe that for $d > 4$ one has $1 < s_c$ and thus the construction of weak solutions works for data of supercritical regularity with respect to the scaling of the equation. However it requires finite energy of the initial data. The main goal of this paper is to show that weak solutions still exist for infinite energy, almost surely with respect to a large class of probability measures.

Let us now describe precisely the initial data sets (statistical ensembles) that we shall consider in this article. Here we follow [10]. Let $0 < s < 1$ and let $(v_0, v_1) \in \mathcal{H}^s$ with Fourier series

$$v_j(x) = a_j + \sum_{n \in \mathbb{Z}_*^d} (b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x)), \quad j = 0, 1,$$

where $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$. Then let $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$, $n \in \mathbb{Z}_*^d$, $j = 0, 1$ be a sequence of independent real random variables given on a probability space $(\Omega, \mathcal{F}, \mathbf{p})$ with a joint distribution θ satisfying

$$\exists c > 0, \quad \forall \gamma \in \mathbb{R}, \quad \int_{-\infty}^{\infty} e^{\gamma x} d\theta(x) \leq e^{c\gamma^2}.$$

We then define the random variables v_j^ω by

$$v_j^\omega(x) = \alpha_j(\omega)a_j + \sum_{n \in \mathbb{Z}_*^d} (\beta_{n,j}(\omega)b_{n,j} \cos(n \cdot x) + \gamma_{n,j}(\omega)c_{n,j} \sin(n \cdot x)),$$

and we define the measure $\mu_{(v_0, v_1)}$ on \mathcal{H}^s as the image of \mathbf{p} under the map

$$\omega \longmapsto (v_0^\omega, v_1^\omega) \in \mathcal{H}^s.$$

We then define \mathcal{M}^s by

$$\mathcal{M}^s = \bigcup_{(v_0, v_1) \in \mathcal{H}^s} \{\mu_{(v_0, v_1)}\}.$$

For $(u_0, u_1) \in \mathcal{H}^s$, denote by

$$(1.2) \quad S(t)(u_0, u_1) = \cos(t\sqrt{-\Delta})(u_0) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(u_1),$$

the free wave evolution. Then our result reads

THEOREM 1.1. — *Let $d \geq 4$, $0 < s < 1$ and $\mu = \mu_{(v_0, v_1)} \in \mathcal{M}^s$. Then there exists a set Σ of full μ measure so that for every $(u_0, u_1) \in \Sigma \subset \mathcal{H}^s$ the equation (1.1) with initial condition $(u(0), \partial_t u(0)) = (u_0, u_1)$ has a solution*

$$u(t) = S(t)(u_0, u_1) + w(t),$$