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A PRESENTATION FOR THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

BY LUIS PARIS & BŁAŻEJ SZEPIETOWSKI

ABSTRACT. — Let $N_{g,n}$ denote the nonorientable surface of genus g with n boundary components and $\mathcal{M}(N_{g,n})$ its mapping class group. We obtain an explicit finite presentation of $\mathcal{M}(N_{g,n})$ for $n \in \{0,1\}$ and all g such that $g+n>3$.

RÉSUMÉ (*Une présentation du mapping class groupe d'une surface non orientable*)
Notons $N_{g,n}$ la surface non orientable de genre g avec n composantes de bord et $\mathcal{M}(N_{g,n})$ son mapping class groupe. On obtient une présentation finie explicite de $\mathcal{M}(N_{g,n})$ pour $n \in \{0,1\}$ et pour tout g tel que $g+n>3$.

1. Introduction

Let F be a compact connected surface with (possibly empty) boundary and let $\mathcal{P} = \mathcal{P}_m = \{P_1, \dots, P_m\}$ be a set of m distinguished points in the interior of F, called *punctures*. We define $\mathcal{H}(F, \mathcal{P})$ to be the group of all, orientation preserving if F is orientable, homeomorphisms $h \colon F \to F$ such that $h(\mathcal{P}) = \mathcal{P}$ and h is equal to the identity on the boundary of F. The *mapping class group* $\mathcal{M}(F, \mathcal{P})$ of F relatively to $\mathcal{P}$ is the group of isotopy classes of elements of $\mathcal{H}(F, \mathcal{P})$. The *pure mapping class group* $\mathcal{PM}(F, \mathcal{P})$ is the subgroup of $\mathcal{M}(F, \mathcal{P})$

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consisting of the isotopy classes of homeomorphisms fixing each puncture. If $\mathscr{P} = \varnothing$ then we drop it in the notation and write simply $\mathscr{M}(F)$. If $\mathscr{P} = \{P\}$ then we write $\mathscr{M}(F, P)$ instead of $\mathscr{M}(F, \{P\})$. A compact surface of genus g with n boundary components will be denoted by $S_{g,n}$ if it is orientable, or by $N_{g,n}$ if it is nonorientable.

Historically, McCool [20] gave the first algorithm for finding a finite presentation for $\mathscr{M}(S_{g,1})$ for any g. His approach is purely algebraic and no explicit presentation has been derived from this algorithm. In their ground breaking paper [13] Hatcher and Thurston gave an algorithm for computing a finite presentation for $\mathscr{M}(S_{g,1})$ from its action on a simply connected simplicial complex, the *cut system complex*. By this algorithm, Harer [10] obtained a finite, but very unwieldy, presentation for $\mathscr{M}(S_{g,1})$ for any g. This presentation was simplified by Wajnryb [29, 30], who also gave a presentation for $\mathscr{M}(S_{g,0})$. Using Wajnryb's result, Matsumoto [19] found other presentations for $\mathscr{M}(S_{g,1})$ and $\mathscr{M}(S_{g,0})$, and Gervais [9] found a presentation for $\mathscr{M}(S_{g,n})$ for arbitrary $g \geq 1$ and n. Starting from Matsumoto's presentations, Labruère and Paris [16] computed a presentation for $\mathscr{M}(S_{g,n}, \mathscr{P}_m)$ for arbitrary $g \geq 1$, n and m. Benvenuti [1] and Hirose [14] independently recovered the Gervais presentation from the action of $\mathscr{M}(S_{g,n})$ on two different variations of the Harvey's curve complex [11], instead of the cut system complex.

Until present, finite presentations of $\mathscr{M}(N_{g,n}, \mathscr{P}_m)$ were known only for a few small values of (g, n, m), with $g \leq 4$. Using results of Lickorish [17, 18], Chillingworth [7] found a finite generating set for $\mathscr{M}(N_{g,0})$ for arbitrary g. This set was extended for $m > 0$ by Korkmaz [15], and for $n + m > 0$ and $g \geq 3$ by Stukow [23]. For every nonorientable surface $N_{g,n}$ there is a covering $p\colon S_{g-1,2n} \to N_{g,n}$ of degree two. By a result of Birman and Chillingworth [4], generalized for $n > 0$ in [26], $\mathscr{M}(N_{g,n})$ is isomorphic to the subgroup of $\mathscr{M}(S_{g-1,2n})$ consisting of elements commuting with the covering involution. However, since the image of $\mathscr{M}(N_{g,n})$ has infinite index in $\mathscr{M}(S_{g-1,2n})$, it seems that it would be very hard to obtain a finite presentation for $\mathscr{M}(N_{g,n})$ from a presentation of $\mathscr{M}(S_{g-1,2n})$. In [24] an algorithm for finding a finite presentation for $\mathscr{M}(N_{g,n})$ for any g and n is given, based on a result of Brown [6] and the action of $\mathscr{M}(N_{g,n})$ on the curve complex (following the idea of [1]). By this algorithm, an explicit finite presentation for $\mathscr{M}(N_{4,0})$ was obtained in [25].

In this paper we apply the algorithm given in [24] to find an explicit finite presentation for $\mathscr{M}(N_{g,n})$ for $n \in \{0, 1\}$ and all g such that $g + n > 3$. We present $\mathscr{M}(N_{g,1})$ as a quotient of the free product $\mathscr{M}(S_{\rho,r}) * \mathscr{M}(S_{0,1}, \mathscr{P}_g)$, where $g = 2\rho + r$ and $r \in \{1, 2\}$. The factor $\mathscr{M}(S_{\rho,r})$ comes from an embedding of $S_{\rho,r}$ in $N_{g,1}$ and it is generated by Dehn twists. The factor $\mathscr{M}(S_{0,1}, \mathscr{P}_g)$, which is isomorphic to the braid group, comes from the embedding $\mathscr{M}(S_{0,1}, \mathscr{P}_g) \to \mathscr{M}(N_{g,1})$ defined in [26], and it is generated by $g - 1$ crosscap transpositions. There are three families of defining relations of $\mathcal{M}(N_{g,1})$: (A) relations from $\mathcal{M}(S_{\rho,r})$ between Dehn twists, (B) braid relations between crosscap transpositions, and (C) relations involving generators of both types. A presentation for $\mathcal{M}(N_{g,0})$ is obtained from that of $\mathcal{M}(N_{g,1})$ by adding three relations.

The presentations for $\mathcal{M}(N_{g,1})$ and $\mathcal{M}(N_{g,0})$ are given respectively in Theorems 3.5 and 3.6 in Section 3. They are proved simultaneously by induction on g. The base cases $(g, n) \in \{(3, 1), (4, 0)\}$ are proved in Section 4. Theorem 3.5 is proved in Section 7 under the assumption that Theorem 3.6 is true. The proof of Theorem 3.6 uses the action of $\mathcal{M}(N_{g,0})$ on the ordered complex of curves defined in Section 5, and it occupies Sections 6, 8, where presentations of stabilizers of vertices are calculated, and Sections 9, 10, where we deal with relations corresponding to simplices of dimensions 1 and 2.

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2. Preliminaries

2.1. Simple closed curves and Dehn twists. — By a *simple closed curve* in F we mean an embedding $\gamma\colon S^1 \to F \backslash \partial F$. Note that γ has an orientation; the curve with the opposite orientation but same image will be denoted by γ^{-1}. By abuse of notation, we will often identify a simple closed curve with its oriented image and also with its isotopy class. We say that γ is *generic* if it does not bound a disc nor a Möbius band and is not isotopic to a boundary component. According to whether a regular neighborhood of γ is an annulus or a Möbius strip, we call γ respectively *two-* or *one-sided.* We say that γ is *nonseparating* if $F \backslash \gamma$ is connected and *separating* otherwise.

Given a two-sided simple closed curve γ, T_γ denotes a Dehn twist about γ. On a nonorientable surface it is impossible to distinguish between right and left twists, so the direction of a twist T_γ has to be specified for each curve γ. In this paper it is usually indicated by arrows in a figure. Equivalently we may choose an orientation of a regular neighborhood of γ. Then T_γ denotes the right Dehn twist with respect to the chosen orientation. Recall that T_γ does not depend on the orientation of γ.