PANORAMAS ET SYNTHÈSES 14

WHY ARE BRAIDS ORDERABLE?

Patrick Dehornoy
Ivan Dynnikov
Dale Rolfsen
Bert Wiest

Société Mathématique de France 2002
Publié avec le concours du Centre National de la Recherche Scientifique
et du Ministère de la Culture et de la Communication

P. Dehornoy

Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139,
Université de Caen, 14032 Caen, France.

E-mail : `dehornoy@math.unicaen.fr`

Url : `www.math.unicaen.fr/∼dehornoy`

I. Dynnikov

Dept. of Mechanics and Mathematics Moscow State University; Moscow 119992 GSP-2, Russia.

E-mail : `dynnikov@mech.math.msu.su`

Url : `//mech.math.msu.su/∼dynnikov`

D. Rolfsen

Mathematics Department; Univ. of British Columbia; Vancouver BC, V6T 1Z2 Canada.

E-mail : `rolfsen@math.ubc.ca`

Url : `www.math.ubc.ca/∼rolfsen`

B. Wiest

IRMAR, Université de Rennes 1, Campus Beaulieu, 35042 Rennes, France.

E-mail : `bertw@math.univ-rennes1.fr`

Url : `//name.math.univ-rennes1.fr/bertold.wiest`

2000 *Mathematics Subject Classification.* — 20F36, 06F15, 20C40, 20N02, 57M60, 57M07.

Key words and phrases. — Braid groups; ordered groups; mapping class groups; self-distributive systems; finite trees; well-orderings; free group automorphisms; automatic groups; laminations; hyperbolic geometry.

The work of Ivan Dynnikov was supported by RFBR (grant # 99-01-00090).

WHY ARE BRAIDS ORDERABLE?

Patrick Dehornoy, Ivan Dynnikov, Dale Rolfsen, Bert Wiest

***Abstract.* —** In the decade since the discovery that Artin's braid groups enjoy a left-invariant linear ordering, several quite different approaches have been applied to understand this phenomenon. This book is an account of those approaches, involving self-distributive algebra, uniform finite trees, combinatorial group theory, mapping class groups, laminations, and hyperbolic geometry.

***Résumé* (Pourquoi les tresses sont-elles ordonnables ?). —** Environ dix ans ont passé depuis la découverte du caractère ordonnable des groupes de tresses, et des méthodes diverses ont été proposées pour expliquer le phénomène. Le but de ce texte est de présenter ces approches variées, qui mettent en jeu l'algèbre auto-distributive, les arbres finis, la théorie combinatoire des groupes, les groupes de difféomorphismes, la théorie des laminations, et la géométrie hyperbolique.

Ein Jahrzehnt ist vergangen seit der Entdeckung, dass Artins Zopfgruppen eine links-invariante Ordnung besitzen, und verschiedene Methoden wurden in der Zwischenzeit vorgeschlagen, um zu einem tieferen Verständnis dieses Phänomens zu gelangen. Ziel dieses Buches ist es, ein Resümee dieser Techniken zu geben. Selbstdistributive Algebren, endliche Bäume, kombinatorische Gruppentheorie, Abbildungsklassengruppen, Laminationen, und hyperbolische Geometrie kommen dabei zusammen.

За десять лет, прошедшие после открытия, что артиновские группы кос обладают левоинвариантным линейным порядком, возник целый ряд различных подходов для объяснения этого явления. Данная книга посвящена описанию этих подходов, которые основаны на самодистрибутивных операциях, теории однородных конечных деревьев, комбинаторной теории групп, группах классов отображений, ламинациях и гиперболической геометрии.

CONTENTS

INTRODUCTION

An idea whose time was overdue.

This book is about braids and orderings. The braid groups B_n were introduced by Emil Artin [**2**] in 1925 (see also [**3**]) and have been studied intensively ever since [**17, 20**]. Indeed, many of the ideas date back to the 19th century, in the works of Hurwicz, Klein, Poincaré, Riemann, and certainly other authors. One can even find a braid sketched in the notebooks of Gauss [**66**]—see [**123**] for a discussion about Gauss and braids, including a reproduction of the picture he drew in his notebook. The n-strand braid group B_n has the well-known presentation (other definitions will be given later):

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1}\,;\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1\rangle.$$

We use B_n^+ for the monoid with the above presentation, which is called the n-strand braid monoid. To each braid, there is an associated permutation of the set $\{1, \ldots, n\}$, with $\sigma_i \mapsto (i, i+1)$, defining a homomorphism $B_n \to \mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the symmetric group on n objects. The kernel of this mapping is the *pure* braid group P_n.

The theory of ordered groups is also well over a hundred years old. One of the basic theorems of the subject is Hölder's theorem, published in 1902 [**75**], that characterizes the additive reals as the unique maximal Archimedian ordered group. It is remarkable, and somewhat puzzling, that it has taken so long for these two venerable subjects to come together as they now have.

A group or a monoid G is *left-orderable* if there exists a strict linear ordering $<$ of its elements which is left-invariant: $g < h$ implies $fg < fh$ for all f, g, h in G. If, in addition, the ordering is a well-ordering, we say that G is *left-well-orderable*. If there is an ordering of G which is invariant under multiplication on both sides, we say that G is *orderable*, or for emphasis, *bi-orderable*. This book is devoted to explaining the following results, discovered within the last decade:

Theorem I.1. — *The Artin braid group B_n is left-orderable, by an ordering which is a well-ordering when restricted to B_n^+.*

Theorem I.2. — *The pure braid group P_n is bi-orderable, by an ordering which is a well-ordering when restricted to $P_n \cap B_n^+$.*

The braid groups have been an exceptionally active mathematical subject in recent decades. The field exploded in the mid 1980's with the revolutionary discoveries of Vaughan Jones [**78**], providing strong connections with operator theory, statistical mechanics and other notions of mathematical physics. Great strides have been made in recent years in understanding the very rich representation theory of the braid groups. The classical Burau representation [**13**] was shown to be unfaithful [**108**], [**97**]. The long-standing question of whether the braid groups are linear (isomorphic to finite-dimensional matrix groups) was recently answered in the affirmative, by Daan Krammer [**84, 85**] and Stephen Bigelow [**6**].

Despite the high degree of interest in braid theory, the importance of the left-orderability of the braid groups, announced in 1992 [**33**], was not widely recognized at first. A possible explanation for this is that the methods of proof were rather unfamiliar to most topologists, the people most interested in braid theory. As will be seen in Chapter 2, that proof involves rather delicate combinatorial and algebraic constructions, which were partly motivated by (while being logically independent of) questions in set theory — see [**81**] for a good introduction. Subsequent combinatorial work brought new results and proposed new approaches: David Larue established in [**88, 87**] results anticipating those of [**59**], Richard Laver proved in [**93**] that the restriction of the braid ordering to B_n^+ is a well-ordering (presumably the deepest result known so far about the braid order), Serge Burckel gave an effective version of the latter result in [**14, 15**]. However, these results were also not widely known for several years.

The challenge of finding a topological proof of left-orderability of B_n led to the five-author paper [**59**], giving a completely different construction of an ordering of B_n as a mapping class group. Remarkably, it leads to exactly the same ordering as [**33**]. Soon after, a new technique [**130**] was applied to yield yet another proof of orderability of the braid groups (and many other mapping class groups), using ideas of hyperbolic geometry, and moreover giving rise to many possible orderings of the braid groups. This argument, pointed out by William Thurston, uses ideas of Nielsen [**111**] from the 1920's. It is interesting to speculate whether Nielsen himself might have solved the problem, if asked whether braid groups are left-orderable in the following language: Does the mapping class group of an n-punctured disk act effectively on the real line by order-preserving homeomorphisms? Nielsen had laid all the groundwork for an affirmative answer.

More recently, a new topological approach using laminations was proposed, one that is also connected with the Mosher normal form based on triangulations [**109**]. Also, a combinatorial interpretation of the results of [**130**] was proposed by Jonathon Funk in [**64**], including a connection with the theory of topoi.

The braid groups are known to be automatic [**137**]. Without burdening the reader with technical details, it should be mentioned that the ordering of B_n and certain other surface mapping class groups (nonempty boundary) can be considered automatic as well, meaning roughly that it may be determined by some finite-state automaton [**128**].

Theorem I.2 appeared in [**82**], and it relies on a completely different approach, namely using the Magnus representation of a free group. Subsequent work [**127**] has shown how different general braid groups B_n and the pure braid groups P_n are from the point of view of orderability: in particular, for $n \geqslant 5$, the group B_n is not locally indicable, which implies that it is not bi-orderable in a strong sense, namely that no left-ordering of B_n can bi-order a subgroup of finite index, such as P_n [**125**].

The importance of being orderable

As will be recalled in Chapter 1, the orderability of a group implies various structural consequences about that group and derived objects. The fact that B_n is left-orderable implies that it is torsion-free, which had been well known. However, it also implies that the group ring $\mathbb{Z}B_n$ has no zero-divisors, which was a natural open question. Biorderability of P_n shows that $\mathbb{Z}P_n$ embeds in a skew field. In addition, it easily implies that the group P_n has unique roots, a result proved in [**4**] by complicated combinatorial arguments, and definitely not true for B_n.

One may argue that such general results did not dramatically change our understanding of braid groups. The main point of interest, however, is not—or not only—the mere existence of orderings on braid groups, but the particular nature and variety of the constructions we shall present. Witness the beautiful way the order on P_n is deduced from the Magnus expansion in Chapter 9, the fascinating connection between the uncountable family of orderings on B_n constructed in Chapter 7 and the Nielsen–Thurston theory, and, chiefly, the specific properties of one particular ordering on B_n. Here we refer to the ordering of B_n sometimes called the Dehornoy ordering in literature, which will be called the σ-ordering in this text.

Typically, it is the specific form of the braids greater than 1 in the σ-ordering that led to the new, efficient algorithm for the classical braid isotopy problem described in Chapter 3, and motivated the further study of the algorithms described in Chapters 6 and 8. But what appears to be of the greatest interest here is the remarkable convergence of many approaches to one and the same object: at least six different points of view end up today with the σ-ordering of braids, and this, in our opinion, is the main hint that this object has an intrinsic interest. Just to let the reader feel the flavour of some of the results, we state below various characterizations of the σ-ordering—the terms will be defined in the appropriate place. So, the braid β_1 is smaller than the braid β_2 in the σ-ordering if and only if

- (*in terms of braid words*) the braid $\beta_1^{-1}\beta_2$ has a braid word representative where the generator σ_i with smallest index i appears only positively (no σ_i^{-1});
- (*in terms of action on self-distributive systems*) for some/any ordered LD-system $(S, *, <)$, and for some/any sequence $\vec{x}$ in S, we have $\vec{x} \cdot \beta_1 <^{\mathrm{Lex}} \vec{x} \cdot \beta_2$;

- (*in terms of braid words combinatorics*) any sequence of handle reductions from any braid word representing $\beta_1^{-1}\beta_2$ ends up with a σ-positive word;
- (*in terms of trees, assuming β_1 and β_2 to be positive braids*) the irreducible uniform tree associated with β_1 is smaller than the one associated with β_2;
- (*in terms of automorphisms of a free group*) for some i, the automorphism associated with $\beta_1^{-1}\beta_2$ maps x_j to x_j for $j < i$, and it maps x_i to a word that ends with x_i^{-1};
- (*in terms of mapping class groups*) the standardized curve diagram associated with β_1 first diverges from the one associated with β_2 towards the left;
- (*in terms of hyperbolic geometry*) the endpoint of the lifting of $\beta_1(\gamma_a)$ is larger (as a real number) than the endpoint of the lifting of $\beta_2(\gamma_a)$;
- (*in terms of free group ordering*) we have $\beta_1 \cdot z \lhd \beta_2 \cdot z$ in $\widehat{F_\infty}$;
- (*in terms of Mosher's normal form*) the last flip in the normal form of $\beta_2^{-1}\beta_1$ occurs in the upper half-plane.
- (*in terms of laminations*) the first nonzero coefficient of odd index in the sequence $\beta_1^{-1}\beta_2 \cdot (0, 1, \ldots, 0, 1)$ is positive.

Even if the various constructions of the σ-ordering depend on choosing a particular family of generators for the braid groups, namely the Artin generators σ_i, this convergence might suggest to call this ordering canonical or, at least, standard. This convergence is the very subject of this text: our aim here is not to give a complete study of any of the different approaches—so, in particular, our point of view is quite different from that of [**39**] which more or less exhausts the combinatorial approaches—but to try to let the reader feel the flavour of these different approaches. More precisely—and with the exceptions of Chapter 7 which deals with more general orderings, and of Chapter 9 which deals with ordering pure braids—our aim will be to describe the σ-ordering of braids in the various possible frameworks: algebraic, combinatorial, topological, geometric, and to see which properties can be established by each technique. As explained in Chapter 1, exactly three properties of braids, called **A**, **C**, and **S** here, are crucial to prove that the σ-ordering exists and to establish its main properties. Roughly speaking, each chapter of the subsequent text (except Chapters 7 and 9) will describe one possible approach to the question of ordering the braids, and, in each case, explain which of the properties **A**, **C**, and **S** can be proved: some approaches are relevant for establishing all three properties, while others enable us only to prove one or two of them, possibly assuming some other one already proved.

Organization of the text

Various equivalent definitions of the braid groups are described in Chapter 1, which also includes a general discussion of orderable groups and their rather special algebraic properties. The well-ordering of B_n^+ is also introduced in this chapter.

The remaining chapters contain various approaches to the orderability phenomenon. The combinatorial approaches are gathered in Chapters 2 to 5, while the topological approaches are presented in Chapters 6 to 8.

Chapter 2 introduces left self-distributive algebraic systems (LD-systems) and the action of braids upon such systems. This is the technique whereby the orderability of braids was first demonstrated and the σ-ordering introduced. The chapter sketches a self contained proof of left-orderability of B_n, by establishing Properties **A**, **C** (actually details are given only for its weak variant $\mathbf{C}_\infty$) and **S** with arguments utilizing LD-systems. Here we consider colourings of the strands of the braids, and observe that the braid relations dictate the self-distributive law among the colours. Then we can order braids by choosing orderable LD-systems as colours, a simple idea yet the existence of an orderable LD-system requires an indirect argument. The chapter concludes with a discussion of the historical origins of orderable LD-systems, which arise in the study of elementary embeddings in the foundations of set theory.

A combinatorial algorithm called handle reduction is the subject of Chapter 3. This procedure, which extends the idea of word reduction in a free group, is a very efficient procedure in practice for determining whether a braid word represents a braid larger than 1, and incidentally gives a rapid solution to the word problem in the braid groups. Handle reduction gives an alternative proof of Property **C**, under the assumption that Property **A** holds.

Another combinatorial technique, due to Serge Burckel, is to encode positive braid words by finite trees. This is the subject of Chapter 4, in which one proves that the restriction of the σ-ordering to B_n^+ is a well-ordering by considering a natural ordering of the associated trees and using a tricky transfinite induction. This approach provides arguments for Properties **C** and **S**, however assuming (as with handle reduction) Property **A**. The advantage of the method is that it assigns a well-defined ordinal to each braid in B_n^+. By using a variant of the σ-ordering, one obtains a well-ordering of B_∞^+.

Chapter 5 contains an approach to the σ-ordering using a very classical fact, that the braid groups can be realized as a certain group of automorphisms of a free group. As observed by David Larue, this method yields a quick proof of Property **A**, an alternative proof of Property **C** (hence an independent proof of left-orderability of B_∞) and a simple criterion for recognizing whether a braid is σ-positive, in terms of its action on the free group.

We begin the topological description of the σ-ordering in Chapter 6. Here we realize B_n as the mapping class group of a disk with n punctures. The braid action can be visualized by use of curve diagrams which provide a canonical form for the image of the real line, if the disk is regarded as the unit complex disk. This was the first geometric argument for the left-orderability of the braid groups, and it is remarkable that the ordering described in this way is identical with the original, *i.e.*, with the σ-ordering. An advantage of this approach is that it also applies to more general mapping class groups. We emphasize that Chapters 5 and 6 are based on very similar ideas, except that the first one is algebraic while the second is more geometric.

The discussion in Chapter 7 interprets braid orderings in terms of Nielsen–Thurston theory. The key observation is that the universal cover of the punctured disk has a natural embedding in the hyperbolic plane. Thereby, braids act on a family of hyperbolic geodesics, which have a natural ordering. This point of view provides an infinitude of inequivalent orderings of braid groups and many other mapping class groups. The σ-ordering on B_n corresponds to choosing a particular geodesic in $\mathbb{H}^2$. We also outline in this chapter the interpretation developed by Jonathon Funk, in which a certain linear ordering of words in the free group is preserved under the braid automorphisms as considered in Chapter 5.

Chapter 8 continues the discussion of the σ-ordering in terms of mapping classes. However, here, the geometric approach is rephrased in combinatorial terms by use of two somewhat different devices involving triangulations. The first was inspired by the technique employed by Lee Mosher to establish that mapping class groups are automatic. It develops a new canonical form for braids and a method for determining σ-ordering by means of a finite state automaton. The second approach, developed in [**53**], uses integral laminations. One encodes the action of a braid on the disk by counting intersections of the image of a certain triangulation with a lamination. This leads to an independent proof of Property **A** and yet another characterization of braids larger than 1 in the σ-ordering.

The final chapter is an account of an ordering of the *pure* braid groups. Unlike the full braid groups, the groups P_n of pure braids can be given an ordering which is invariant under multiplication on both sides. This ordering is defined algebraically, using the Artin combing technique, together with a specific ordering of free groups using the Magnus expansion. By appropriate choice of conventions, this ordering has the property that braids in $P_n \cap B_n^+$ are larger than 1 and well-ordered. The chapter ends with a discussion showing that any two-sided ordering of P_n is necessarily incompatible with every left-ordering of B_n for $n \geqslant 5$.

Guidelines to the reader

There are many sorts of readers who may be interested in this text, with various styles of mathematical understanding. Thus different approaches are bound to appeal to different readers. The reader with a mostly algebraic or combinatorial culture may feel uncomfortable with informal definitions in the geometric constructions of Chapters 6, 7, or 8, while another reader coming from the world of topology or geometry may find Chapter 2 and, even more, Chapter 4 quite mysterious, and lacking conceptual understanding in the case of the latter. It is impossible to claim that one approach is definitely better than another, as every one of them brings some specific result or intuition that is so far inaccessible to the others. An attempt has been made to keep the chapters relatively self-contained; so, apart from Chapter 1, all chapters are parallel one to the other rather than logically interdependent, and, therefore, from Chapter 2, the reader can take the chapters essentially in whatever order he or she likes.

We mentioned that three properties of braids called **A**, **C**, and **S** play a crucial role, and that our main task in this text will be to prove these properties using various possible approaches. In spite of the above general remarks, it might be useful that we propose answers to the question: which of these approaches offers the quickest, or the most elementary, proof of Properties **A**, **C**, and **S**? The answer depends of course on the mathematical preferences of the reader. As for Property **A**, the shortest proofs are the one using the automorphisms of a free group in Chapter 5, and—even shorter once the formulas (8.5.14) have been guessed—the one using laminations in Chapter 8. The argument involving self-distributivity by contrast is more conceptual and naturally connected with orderings, but it is technically quite involved. As for Property **C**, the shortest argument is probably the one involving self-distributivity as outlined in Chapter 2, but one may prefer the approach through handle reduction, which uses nothing exotic and gives an efficient algorithm in addition, or the curve diagram approach of Chapter 6, which gives a less efficient method and requires considerable effort to be made rigorous, but appeals to a natural geometric intuition. Finally, for Property **S**, the hyperbolic geometry argument of Chapter 7 is probably the more interesting one, as it gives the result not only for the σ-ordering, but also for a whole family of different orderings. On the other hand, even if it may appear conceptually less satisfactory in its present exposition, the combinatorial approach of Chapter 4 gives the most precise and effective version of Property **S**.

Although they are conceptually simple, the braid groups are very subtle nonabelian groups which have given up their secrets only reluctantly over the years. They will undoubtedly continue to supply us with surprises and fascination, and so will in particular their orderings: despite the many approaches and results mentioned in the text below, a lot of questions about braid orderings remain open today, and further developments can be expected. For the moment, we hope that this small text, which involves techniques of algebra, combinatorics, hyperbolic geometry, topology, and has even a loose connection with set theory, can illuminate some facets of the question, "Why are braids orderable?"

The preparation of this text was coordinated by P.D.; Chapters 2 to 4 have been mostly written by P.D., Chapters 6 and 7 by B.W., Chapter 8 by I.D., and Chapter 9 by D.R.; the other chapters are common work of two or more authors.

We thank all colleagues and friends who have suggested corrections or improvements, specially Roger Fenn and Christian Kassel. We also thank Hervé Sibert for his help in practical implementations, and the referees for their valuable suggestions.

CHAPTER 1

A LINEAR ORDERING OF BRAIDS

In this chapter, we briefly recall some of the standard definitions of the braid groups. Then we introduce three basic properties of braids, called **A**, **C**, and **S** in the sequel, and we show how they allow us to define a linear ordering of braids that is compatible with multiplication on one side. Finally, we describe some general properties of this ordering.

1.1. Braid groups

As mentioned before, the Artin braid group on n strands, denoted by B_n, is defined by the presentation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \,;\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1 \rangle.$$

The braid group on infinitely many strands, denoted B_∞, is defined by a presentation with infinitely many generators $\sigma_1, \sigma_2, \ldots$ subject to the same relations.

The aim of this section is to show how the groups B_n arise in several different ways as special cases of some natural mathematical objects in geometry and algebra.

1.1.1. Isotopy classes of braid diagrams. — Let D^2 be the unit disk with centre 0 in the complex plane $\mathbb{C}$, and let D_n be the disk D^2 with n regularly spaced points in the real axis as distinguished points; we call these points the puncture points of D^2.

Definition 1.1.1. — We define an *n-strand geometric braid* to be the image of an embedding b of the disjoint union $\coprod_{j=1}^{n}[0_j, 1_j]$ of n copies of the interval $[0,1]$ into the cylinder $[0,1]\times D^2$ satisfying the following properties: (i) for t in $[0_j, 1_j]$, the point $b(t)$ lies in $\{t\} \times D^2$; (ii) the set $\{b(0_1), \ldots, b(0_n)\}$ is the set of punctures of $\{0\} \times D^2$, and similarly the set $\{b(1_1), \ldots, b(1_n)\}$ is the set of punctures of $\{1\} \times D^2$.

The image of each interval is called a strand of the braid; the idea is that we have n strands running continuously from left to right (visualizing the unit interval as being