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ON THE GENESIS OF THE CONCEPT OF COVARIANT DIFFERENTIATION

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ABSTRACT. — The purpose of this paper is to reconsider the genesis of the concept of covariant differentiation, which is interpreted as arising out of two traditions running through 19th-century research work. While the first tradition, of an algebraic nature, was responsible for the "algorithmic" emergence of the concept, the second, analytical in character, was essentially concerned with the import of covariant differentiation as a broader kind of differentiation. The methodological contrast that these two traditions exhibit, concerning the use of algebraic and variational methods, was mainly evidenced in Ricci-Curbastro's work, and was a significant factor in the genesis of tensor analysis. The emergence of the notion of covariant differentiation in his research work may, indeed, be interpreted as the resolution of that methodological contrast into the definitive form of a conceptual synthesis.

RÉSUMÉ. — SUR L'ORIGINE DU CONCEPT DE DÉRIVATION COVARIANTE. Cet article se propose d'interpréter l'origine du concept de dérivation covariante comme conséquence de deux traditions de recherche au XIX^{e} siècle. Alors que la première tradition, de nature algébrique, est à l'origine de l'émergence «*algorithmique*» du concept, la seconde, de caractère analytique, se rapporte essentiellement à la signification de la dérivation covariante comme extension ou généralisation de la dérivation usuelle. L'opposition méthodologique que manifestent ces deux traditions, à propos de l'utilisation de méthodes algébriques ou variationnelles, apparaît principalement dans l'œuvre de Ricci-Curbastro, et fut un facteur fondamental dans la genèse de l'analyse tensorielle. L'émergence de la notion de dérivation covariante dans son travail de recherche peut, de fait, être interprétée comme la résolution de cette opposition méthodologique sous la forme décisive d'une synthèse conceptuelle.

1. INTRODUCTION

Emerging at the end of the 19th century with the work of the Italian

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mathematician G. Ricci-Curbastro, absolute differential calculus (and subsequently tensor analysis) appears historically as one of the most important links between Riemann's concept of space and the relativistic theory of gravitation. As an extension of the usual calculus to general geometrical contexts, this theory indeed represents one of the most important developments of Riemann's geometrical conceptions in the latter part of the 19th century. On the other hand, with reference to such notions as that of covariant differentiation and of the tensor, the theory set out by the Italian mathematician pointed to the possibility of an invariant formulation of analytical problems, possibly of a physical nature: a technical possibility which was to play a leading role in the mathematical expression of Einstein's ideas, some decades later.

Bearing that in mind, the aim of this paper is to provide a reconstruction of the emergence of the first fundamental concept of absolute differential calculus — that of covariant differentiation — as marking the convergence of various research traditions in mathematical thought, prevailing in the 19th century. More specifically, this reconstruction is based on a number of historiographical tenets, concerning different instances of the impact of the idea of invariance, which I shall now detail.

First of all, one may hold that there was an "algorithmic" genesis of the concept of covariant differentiation, arising out of a purely algebraic research tradition. As already suggested by other authors,[1] prior to Ricci-Curbastro's work, this concept had originated in Christoffel's approach, as the result of a research tradition, consisting in the application of the methods of the theory of algebraic invariants to analytical matters. In this context, the algorithm of covariant differentiation was used by Christoffel as a well-defined technique in a particular field of research, that of differential quadratic forms: in particular, it had the specific function of allowing a general programme to be carried out, that of the "reduction" of the theory of differential invariants to that of algebraic forms. As we shall see, this research tradition had clearly exerted an influence on Ricci-Curbastro in a period before his work was directly concerned with the creation of the absolute differential calculus. Such a methodological influence is no chance feature: as we shall see, an embryonic form of the algebraic research

[1] In particular, see the recent book [Reich 1994].

tradition on differential invariants was already at work in the mathematical community of post-Unification Italy, with the geometrical work of Casorati.

On the other hand, despite the fundamental significance of the algebraic tradition with respect to differential invariants, it is tenable that the conceptual origin of covariant differentiation — as a generalisation of the usual differentiation — was independent of that tradition. The appearance of a Riemannian differentiation, indeed, finds its true justification only when one takes into account the emergence of a second research tradition, which to some extent ran counter to the former from a methodological point of view. More specifically, this second tradition was concerned with a close investigation of "differential parameters", as arising out of the work of the French mathematician G. Lamé and developed mainly through the research work of E. Beltrami. This new tradition made its presence felt in the process leading to the emergence of absolute differential calculus, most recognisably when Christoffel's research programme was extended by Ricci-Curbastro to the study of differential parameters.

The point that needs to be emphasised is the contrast inherent in such a switch in topics of investigation. At that time, indeed, the research tradition concerned with differential parameters was grounded, methodologically speaking, on the use of the calculus of variations and only partly on algebraic methods. This was no chance feature, since this second tradition was closely connected to the thrust of classical mathematical physics, and hence to the study of partial differential equations. As we shall see, Ricci-Curbastro effected the introduction of the concept of covariant differentiation precisely for the purposes of furthering the study of differential equations, his aim being to arrive at an invariant expression of these equations in order to simplify their investigation. It is this very cross-over of the contexts of interpretation and methods — i.e., to use modern terminology, the analytical interpretation of an algebraic technique introduced to tackle some analytical problems — that warranted the emergence of the concept of covariant differentiation.

Thus, the emergence and the very genesis of the concept of covariant differentiation appears as a specific synthesis of many research traditions concerning the idea of invariance, running through the 19th century: differential invariants, differential parameters and algebraic invariants. This fact — which, indeed, means that absolute differential calculus, together with Klein's "Erlangen programme", represented one of the most significant products of the idea of invariance in 19th-century mathematical thought — was especially significant with regard to the physical aspects of invariance that were to emerge with general relativity.

And, as a final point, reconstruction of the genesis of the concept of covariant differentiation makes it possible, *post factum*, to examine the specific features of Ricci-Curbastro's scientific work and, more generally, of the Italian mathematical community's contribution as specific contexts for the appearance of absolute differential calculus. In effect, from a strictly historical point of view, one may view the present paper as a comparative study of some aspects of Ricci-Curbastro's work in differential geometry.

2. RESEARCH TRENDS IN THE 19TH-CENTURY THEORY OF DIFFERENTIAL INVARIANTS

As is well known, the context of research in which Ricci-Curbastro's analytical methods originated was provided by the theory of differential invariants, i.e. the study of differential quantities that are invariant with respect to any particular transformation of coordinates.[2] In this manner, the Italian mathematician's work may be considered as an aspect of a more general phenomenon — the pervasiveness of the idea of invariance — which was a characteristic feature of a large part of mathematics throughout the 19th century [Bell 1945, chap. 20].

In this general context — where concepts of geometrical and algebraic invariance were coming to the fore — the study of differential invariants reflected various analytical requirements associated with the idea of invariance. Indeed, the modern theory of differential invariants reached its unified form only at the beginning of the 20th century,[3] as the outcome of many research traditions at work in the course of the 19th century.

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[2] According to M. Kline, tensor analysis "*is actually no more than a variation on an old theme, namely, the study of differential invariants associated primarily with a Riemannian geometry*" [Kline 1972, p. 1122]. On this subject see also [Reich 1994, 4.1.2.1], [Tonolo 1954, pp. 2–6].

[3] This may be considered to be a result of Klein's thought. See [Veblen 1927, p. 15].

Apart from the approach of G. Halphen [1878] and S. Lie [1884] — which emerged much later in the century — there are essentially two theoretical thrusts which were of major importance in this field.[4]

The first direction — which will be referred to here as the "restricted [or special] theory of differential invariants" — arose from the context of 19th-century differential geometry. In effect, this was a direct sequel of Gauss's geometrical opinions: according to this tradition, a differential invariant is the analytical reflection of the intrinsic properties of surfaces (such as the line element, curvature, and the angle between two directions on a surface).

At the same time, more general invariants — the so-called "differential parameters" — were being studied by another line of research, arising out of the work of Lamé on the equations of classical mathematical physics. In this context, differential parameters are quantities — such as the Laplacian of a function — by means of which it is possible to show the invariance of specific differential equations, in a well-defined geometrical situation.

For quite some time, these research traditions developed, to a large extent, independently. They pursued similar aims but in different fields of research: intrinsic geometry, on the one hand, and the theory of partial differential equations, on the other. They actually converged only in the post-Riemannian period.

Although exhibiting different concerns and activities, the two thrusts of research into differential invariants shared one common methodological element. Both traditions, indeed, were characterised by the implementation of two distinct technical methodologies: the theory of algebraic forms, on the one hand, and the calculus of variations, on the other. The function of these theoretical methods was operational, involved as they both were in the demonstration of the invariance (with respect to particular transformations of coordinates) of known differential quantities and the search for new, analogous, quantities.

From an operational standpoint, this methodological duality was of no particular significance for the development of the theory of differential invariants: as we shall see later, apart from some particular cases, the

[4] On this subject, see [Reich 1973], [Struik 1933], [Veblen 1927], [Vincensini 1972], [Weitzenböck 1921].