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LOCAL COLLAPSING, ORBIFOLDS, AND GEOMETRIZATION

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LOCAL COLLAPSING, ORBIFOLDS, AND GEOMETRIZATION

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Abstract. **—** This volume has two papers, which can be read separately. The first paper concerns local collapsing in Riemannian geometry. We prove that a three-dimensional compact Riemannian manifold which is locally collapsed, with respect to a lower curvature bound, is a graph manifold. This theorem was stated by Perelman without proof and was used in his proof of the geometrization conjecture. The second paper is about the geometrization of orbifolds. A three-dimensional closed orientable orbifold, which has no bad suborbifolds, is known to have a geometric decomposition from work of Perelman in the manifold case, along with earlier work of Boileau-Leeb-Porti, Boileau-Maillot-Porti, Boileau-Porti, Cooper-Hodgson-Kerckhoff and Thurston. We give a new, logically independent, unified proof of the geometrization of orbifolds, using Ricci flow.

Résumé **(Effondrements locaux, orbifold et géométrisation). —** Ce volume contient deux articles qui peuvent être lus séparément. Le premier concerne des effondrements locaux en géométrie riemannienne. Nous démontrons qu'une variété riemannienne de dimension 3 qui est localement effondrée, relativement à une borne inférieure de la courbure, est un graphe. Ce théorème était énoncé par Perelman sans démonstration et a été utilisé dans sa preuve de la conjecture de géométrisation. Le second article concerne la géométrisation des orbifolds. Un orbifold fermé orientable de dimension 3 qui ne contient pas de mauvais sous-orbifolds admet une décomposition géométrique d'après le travail de Perelman dans le cas des variétés, et d'après les travaux de Boileau-Leeb-Porti, Boileau-Maillot-Porti, Boileau-Porti, Cooper-Hodgson-Kerckhoff et Thurston. Nous donnons une démonstration nouvelle et unique de la géometrisation des orbifolds, *via* le flot de Ricci.

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LOCALLY COLLAPSED 3-MANIFOLDS

by

Bruce Kleiner & John Lott

Abstract. — We prove that a 3-dimensional compact Riemannian manifold which is locally collapsed, with respect to a lower curvature bound, is a graph manifold. This theorem was stated by Perelman and was used in his proof of the geometrization conjecture.

Résumé. — Nous démontrons qu'une variété riemannienne de dimension 3 qui est localement effondrée, relativement à une borne inférieure de la courbure, est un graphe. Ce théorème était énoncé par Perelman sans démonstration et a été utilisé dans sa preuve de la conjecture de géométrisation.

1. Introduction

1.1. Overview. — In this paper we prove that a 3-dimensional Riemannian manifold which is locally collapsed, with respect to a lower curvature bound, is a graph manifold. This result was stated without proof by Perelman in [**24**, Theorem 7.4], where it was used to show that certain collapsed manifolds arising in his proof of the geometrization conjecture are graph manifolds. Our goal is to provide a proof of Perelman's collapsing theorem which is streamlined, self-contained and accessible. Other proofs of Perelman's theorem appear in [**2, 5, 23, 30**].

In the rest of this introduction we state the main result and describe some of the issues involved in proving it. We then give an outline of the proof. We finish by discussing the history of the problem.

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1.2. Statement of results. — We begin by defining an intrinsic local scale function for a Riemannian manifold.

Definition 1.1. — Let M be a complete Riemannian manifold. Given $p \in M$, the *curvature scale* R_p at p is defined as follows. If the connected component of M containing p has nonnegative sectional curvature then $R_p = \infty$. Otherwise, R_p is the (unique) number $r > 0$ such that the infimum of the sectional curvatures on $B(p,r)$ equals $-\frac{1}{r^2}$.

We need one more definition.

Definition 1.2. — Let M be a compact orientable 3-manifold (possibly with boundary). Give M an arbitrary Riemannian metric. We say that M is a *graph manifold* if there is a finite disjoint collection of embedded 2-tori $\{T_j\}$ in the interior of M such that each connected component of the metric completion of $M - \bigcup_j T_j$ is the total space of a circle bundle over a surface (generally with boundary).

For simplicity, in this introduction we state the main theorem in the case of closed manifolds. For the general case of manifolds with boundary, we refer the reader to Theorem 16.1.

***Theorem 1.3* (cf. [24, Theorem 7.4])**. — *Let c_3 denote the volume of the unit ball in $\mathbb{R}^3$ and let $K \geq 10$ be a fixed integer. Fix a function $A : (0,\infty) \to (0,\infty)$. Then there is a $w_0 \in (0, c_3)$ such that the following holds.*

Suppose that (M,g) is a closed orientable Riemannian 3-manifold. Assume in addition that for every $p \in M$,

(1) $\mathrm{vol}(B(p,R_p)) \leq w_0 R_p^3$ *and*

(2) *For every $w' \in [w_0, c_3)$, $k \in [0,K]$, and $r \leq R_p$ such that* $\mathrm{vol}(B(p,r)) \geq w' r^3$, *the inequality*

$$|\nabla^k \operatorname{Rm}| \leq A(w')\, r^{-(k+2)} \tag{1.4}$$

holds in the ball $B(p,r)$.

Then M is a graph manifold.

1.3. Motivation. — Theorem 1.3, or more precisely the version for manifolds with boundary, is essentially the same as Perelman's [**24**, Theorem 7.4]. Either result can be used to complete the Ricci flow proof of Thurston's geometrization conjecture. We explain this in Section 17, following the presentation of Perelman's work in [**21**].

To give a brief explanation, let $(M, g(\cdot))$ be a Ricci flow with surgery whose initial manifold is compact, orientable and three-dimensional. Put $\widehat{g}(t) = \frac{g(t)}{t}$. Let M_t denote the time t manifold. (If t is a surgery time then we take M_t to be the postsurgery manifold.) For any $w > 0$, the Riemannian manifold $(M_t, \widehat{g}(t))$ has a decomposition into a w-thick part and a w-thin part. (Here the terms "thick" and "thin" are suggested by the Margulis thick-thin decomposition but the definition is somewhat different. In the case of hyperbolic manifolds, the two notions are essentially equivalent.) As $t \to \infty$, the w-thick part of $(M_t, \widehat{g}(t))$ approaches the w-thick part of a complete finite-volume Riemannian manifold of constant curvature $-\frac{1}{4}$, whose cusps (if any) are incompressible in M_t. Theorem 1.3 implies that for large t, the w-thin part of M_t is a graph manifold. Since graph manifolds are known to have a geometric decomposition in the sense of Thurston, this proves the geometrization conjecture.

Independent of Ricci flow considerations, Theorem 1.3 fits into the program in Riemannian geometry of understanding which manifolds can collapse. The main geometric assumption in Theorem 1.3 is the first one, which is a local collapsing statement, as we discuss in the next subsection. The second assumption of Theorem 1.3 is more technical in nature. In the application to the geometrization conjecture, the validity of the second assumption essentially arises from the smoothing effect of the Ricci flow equation.

In fact, Theorem 1.3 holds without the second assumption. In order to prove this stronger result, one must use the highly nontrivial Stability Theorem of Perelman [**19, 25**]. As mentioned in [**24**], if one does make the second assumption then one can effectively replace the Stability Theorem by standard C^K-convergence of Riemannian manifolds. Our proof of Theorem 1.3 is set up so that it extends to a proof of the stronger theorem, without the second assumption, provided that one invokes the Stability Theorem in relevant places; see Sections 1.5.7 and 18.

1.4. Aspects of the proof. — The strategy in proving Theorem 1.3 is to first understand the local geometry and topology of the manifold M. One then glues these local descriptions together to give an explicit decomposition of M that shows it to be a graph manifold. This strategy is common to [**5, 23, 30**] and the present paper. In this subsection we describe the strategy in a bit more detail. Some of the new features of the present paper will be described more fully in Subsection 1.5.

1.4.1. An example. — The following simple example gives a useful illustration of the strategy of the proof.

Let $P \subset H^2$ be a compact convex polygonal domain in the two-dimensional hyperbolic space. Embedding H^2 in the four-dimensional hyperbolic space H^4, let $N_s(P)$ be the metric s-neighborhood around P in H^4. Take M to be the boundary $\partial N_s(C)$, slightly smoothed. If s is sufficiently small then one can check that the hypotheses of Theorem 1.3 are satisfied.

Consider the structure of M when s is small. There is a region $M^{\text{2-stratum}}$, lying at distance $\geq$ const. s from the boundary ∂P, which is the total space of a circle bundle. At scale comparable to s, a suitable neighborhood of a point in $M^{\text{2-stratum}}$ is nearly isometric to a product of a planar region with S^1. There is also a region M^{edge} lying at distance $\leq$ const. s from an edge of P, but away from the vertices of P, which is the total space of a 2-disk bundle. At scale comparable to s, a suitable neighborhood of a point in M^{edge} is nearly isometric to the product of an interval with a 2-disk. Finally, there is a region $M^{\text{0-stratum}}$ lying at distance $\leq$ const. s from the vertices of P. A connected component of $M^{\text{0-stratum}}$ is diffeomorphic to a 3-disk.

We can choose $M^{\text{2-stratum}}$, M^{edge} and $M^{\text{0-stratum}}$ so that there is a decomposition $M = M^{\text{2-stratum}} \cup M^{\text{edge}} \cup M^{\text{0-stratum}}$ with the property that on interfaces, fibration structures are compatible. Now $M^{\text{edge}} \cup M^{\text{0-stratum}}$ is a finite union of 3-disks and $D^2 \times I$'s, which is homeomorphic to a solid torus. Also, $M^{\text{2-stratum}}$ is a circle bundle over a 2-disk, *i.e.*, another solid torus, and $M^{\text{2-stratum}}$ intersects $M^{\text{edge}} \cup M^{\text{0-stratum}}$ in a 2-torus. So using this geometric decomposition, we recognize that M is a graph manifold. (In this case M is obviously diffeomorphic to S^3, being the boundary of a convex set in H^4, and so it is a graph manifold; the point is that one can recognize this using the geometric structure that comes from the local collapsing.)

1.4.2. Local collapsing. — The statement of Theorem 1.3 is in terms of a *local* lower curvature bound, as evidenced by the appearance of the curvature scale R_p. Assumption (1) of Theorem 1.3 can be considered to be a local collapsing statement. (This is in contrast to a global collapsing condition, where one assumes that the sectional curvatures are at least -1 and $\mathrm{vol}(B(p,1)) < \epsilon$ for every $p \in M$.) To clarify the local collapsing statement, we make one more definition.

Definition 1.5. — Let c_3 denote the volume of the Euclidean unit ball in $\mathbb{R}^3$. Fix $\bar{w} \in (0, c_3)$. Given $p \in M$, the *$\bar{w}$-volume scale at p* is

$$r_p(\bar{w}) = \inf \left\{ r > 0 \,:\, \mathrm{vol}(B(p,r)) = \bar{w}\, r^3 \right\}. \tag{1.6}$$

If there is no such r then we say that the $\bar{w}$-volume scale is infinite.

There are two ways to look at hypothesis (1) of Theorem 1.3, at the curvature scale or at the volume scale. Suppose first that we rescale the ball $B(p, R_p)$ to have radius one. Then the resulting ball will have sectional curvature bounded below by -1 and volume bounded above by w_0. As w_0 will be small, we can say that on the curvature scale, the manifold is locally volume collapsed with respect to a lower curvature bound. On the other hand, suppose that we rescale $B(p, r_p(w_0))$ to have radius one. Let $B'(p,1)$ denote the rescaled ball. Then $\mathrm{vol}(B'(p,1)) = w_0$. Hypothesis (1) of Theorem 1.3 implies that there is a big number $\mathcal{R}$ so that the sectional curvature on the radius $\mathcal{R}$-ball $B'(p,\mathcal{R})$ (in the rescaled manifold) is bounded below by $-\frac{1}{\mathcal{R}^2}$. Using this, we deduce that on the volume scale, a large neighborhood of p is well approximated by a large region in a complete nonnegatively curved 3-manifold N_p. This gives a local model for the geometry of M. Furthermore, if w_0 is small then we can say that at the volume scale, the neighborhood of p is close in a coarse sense to a space of dimension less than three.

In order to prove Theorem 1.3, one must first choose on which scale to work. We could work on the curvature scale, or the volume scale, or some intermediate scale (as