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A THEORY OF CHARACTERISTIC CURRENTS ASSOCIATED WITH A SINGULAR CONNECTION

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Abstract

A general theory of characteristic currents associated to singular connections is developed. In particular, a Chern-Weil theory for bundle maps is introduced and systematically studied. This theory generalizes the standard one. It associates to a map $\alpha : E \longrightarrow F$ between bundles with connection, singular "push-forward" and "pullback" connections on E and F respectively. Characteristic classes are then shown to be canonically represented by d-closed currents universally constructed from the "curvature" of these singular connections. When $\text{rank}(E) = \text{rank}(F) = n$ and ϕ is an Ad-invariant polynomial on $\mathfrak{gl}_n$, formulas of the type

$$\phi(\Omega_F) - \phi(\Omega_E) \;=\; \text{Res}_\phi \, \text{Div}(\alpha) + dT$$

are derived, where $\text{Div}(\alpha)$ is a rectifiable current canonically associated to the singular structure of α, where Res_ϕ is a smooth form of classical Chern-Weil type computed as a polynomial in the curvatures Ω_E, Ω_F of E and F, and where T is a canonical, functorial trangression form with coefficients in L^1_{loc}. The cases where E and F are complex or quaternion line bundles are examined in detail, and lead to a new proof of the Riemann-Roch Theorem for vector bundles over algebraic curves.

Applications include: A C^∞-generalization of the Poincaré-Lelong Formula to smooth sections of any smooth vector bundle; Universal formulas for the Thom class as an equivariant characteristic form (i.e., canonical formulas for a de Rham representative of the Thom class of a bundle with connection); A Differentiable Grothendieck-Riemann-Roch Theorem at the level of forms and currents (in both the complex and spin cases). Each of these holds in the general setting of atomic bundle maps, as introduced and studied in [HS]. A variety of formulas relating geometry and characteristic classes are deduced as direct consequences of the theory.

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Introduction

The aim of this paper is to lay the foundations of a theory of Chern-Weil-Simons-type for singular connections on a smooth vector bundle. The notion of a singular connection here is quite general, but the focus will be on certain connections which arise naturally from bundle maps. More specifically, suppose that E and F are smooth vector bundles with connection over a manifold X. Then our theory associates to each homomorphism $\alpha : E \longrightarrow F$ a constellation of d-closed characteristic currents on X defined canonically in terms of the curvature of the bundles and the singularities of the map α. This is essentially a "Chern-Weil Theory for bundle maps" which in the special case where $\alpha \equiv 0$ reduces to the usual construction for the bundles themselves.

The theory is two-sided; one can focus attention either on E or on F (and retrieve, when $\alpha \equiv 0$, the standard theory for E or F). Let us suppose the focus is on E, and fix a **characteristic polynomial** ϕ, i.e., an Ad -invariant polynomial on the Lie algebra of the structure group of E. Standard Chern-Weil Theory associates to ϕ a smooth, closed differential form $\phi(\Omega_E)$ on X which is defined canonically in terms of the curvature of E and which represents a certain characteristic class $\phi(E) \in H^*_{\text{de Rham}}(X)$ determined universally by ϕ. Our theory associates to ϕ a d-closed current $\phi(\Omega_{E,\alpha})$ which is defined in terms of curvature and the singular structure of α, and which also represents $\phi(E)$.

The theory also produces a canonically defined, functorial transgression current $T = T(\phi, \alpha)$ with the property that

$$\phi(\Omega_{E,\alpha}) - \phi(\Omega_E) = dT$$

In the special case where $\text{rank}(E) = \text{rank}(F)$, the characteristic current has the form

$$\phi(\Omega_{E,\alpha}) = \phi(\Omega_F) - S$$

where $\phi(\Omega_F)$ is the standard Chern-Weil form associated to the bundle F and where S is a d-closed current supported on the **singularity set** $\Sigma \equiv \{x \in X : \alpha_x \text{ is not invertible}\}$. Thus S is canonically and functorially cohomologous to $\phi(\Omega_F) - \phi(\Omega_E)$. In many important cases it will turn out that Σ is an oriented submanifold (or more generally an integral cycle) and that S can be written in the form

$$S = \text{Res}_\phi[\Sigma]$$

where Res_ϕ is a smooth differential form on X and $[\Sigma]$ is the integral current determined by integration over Σ. This residue form will be expressed in terms of the curvatures of E and F by a universally determined "residue polynomial". Thus we obtain an equation of currents:

$$\phi(\Omega_F) - \phi(\Omega_E) \;=\; \text{Res}_\phi[\Sigma] + dT.$$

One can think of $\text{Res}_\phi[\Sigma]$ as the characteristic current associated to $\alpha : E \longrightarrow F$ which represents the class $\phi(F) - \phi(E) \in H^*_{\text{de Rham}}(X)$.

Detailed formulas cannot, of course, be derived for arbitrary smooth bundle maps $\alpha : E \longrightarrow F$ since the singularities can be quite pathological. Nevertheless, the theory does apply to quite general maps. By work of the first author and Stephen Semmes we know that under weak assumptions on α, there exist certain associated currents with rectifiability properties, which one might call "Thom-Porteous currents". For generic maps these currents are standard singularity sets defined by rank conditions on α. Our general formulas will often be expressed in terms of smooth "residue" forms paired with these currents.

The analytic assumptions we make on our bundle maps are presented in detail here and certain important properties are established. However, the fundamental results concerning the existence and structure of divisors and more general Thom-Porteous currents appear in [HS].

To give a notion of the nature of the results, we present some elementary but important examples. The first is provided by the case where E and F are complex line bundles over an oriented manifold X. Suppose for simplicity that $\alpha : E \longrightarrow F$ vanishes non-degenerately so that the divisor $\text{Div}(\alpha)$ is the current associated to the oriented codimension-2 submanifold $\Sigma = \{x \in X : \alpha_x = 0\}$. Then for each polynomial $\phi(u) \in \mathbf{R}[u]$ in one indeterminate, we obtain the formula

$$(*) \qquad \phi(f) - \phi(e) = \left\{ \frac{\phi(f) - \phi(e)}{f - e} \right\} \mathrm{Div}(\alpha) + dT$$

where

$$f = \tfrac{i}{2\pi}\Omega_F \quad \text{and} \quad e = \tfrac{i}{2\pi}\Omega_E$$

are the Chern-Weil representatives of the first Chern classes of F and E respectively (and where $\frac{\phi(f)-\phi(e)}{f-e}$ is the obvious polynomial in e and f.) This formula holds in fact for any α which is **atomic**, that is for which

$$\mathrm{tr}(\alpha^{-1}D\alpha) \in L^1_{\mathrm{loc}},$$

(i.e., for which $\mathrm{tr}(\alpha^{-1}D\alpha)$ has an L^1_{loc}-extension across Σ), where D denotes the induced connection on $\mathrm{Hom}(E, F)$. (Under local trivializations of E and F, α is represented by a complex-valued function a, and atomicity is equivalent to the condition $da/a \in L^1_{\mathrm{loc}}$.) The transgression term T in $(*)$ is given by the formula

$$T = \tfrac{i}{2\pi} \left\{ \frac{\phi(f) - \phi(e)}{f - e} \right\} \mathrm{tr}(\alpha^{-1}D\alpha).$$

When $\phi(u) = u$ and E is trivial, equation $(*)$ represents a C^∞ generalization of the classical Poincaré-Lelong formula.

Combining this result with the kernel-calculus of Harvey and Polking [HP] gives a new proof of the Riemann-Roch Theorem for vector bundles over algebraic curves.

Another example of a basic formula coming from the theory is provided by considering a section $\alpha \in \Gamma(V)$ of an even-dimensional vector bundle $V \longrightarrow X$ with spin structure. Assume that α vanishes non-degenerately (or more generally that it is **atomic** in the sense of [HS] below), and let $\mathrm{Div}(\alpha)$ denote its divisor. Now Clifford multiplication by α determines a bundle map $\alpha : \mathcal{S}^+ \longrightarrow \mathcal{S}^-$ between the positive and negative complex spinor bundles canonically associated to V. Consider the function on matrices $\phi(A) = \mathbf{ch}(A) \stackrel{\text{def}}{=} \mathrm{trace}\left\{\exp(\frac{1}{2\pi i}A)\right\}$ which gives the Chern character. Suppose $\mathcal{S}^+$ and $\mathcal{S}^-$ carry connections induced from a riemannian connection on V, and let $\Omega_{\mathcal{S}^\pm}$, Ω_V denote the curvature matrices