Astérisque

PER SALBERGER

# Tamagawa measures on universal torsors and points of bounded height on Fano varieties

*Astérisque*, tome 251 (1998), p. 91-258
<http://www.numdam.org/item?id=AST_1998__251__91_0>

*Astérisque* **251**, 1998, p. 91–258

# TAMAGAWA MEASURES ON UNIVERSAL TORSORS AND POINTS OF BOUNDED HEIGHT ON FANO VARIETIES

*by*

Per Salberger

***Abstract.*** — Let $X$ be a Fano variety over a number field. An Arakelov system of $v$-adic metrics on the anticanonical line bundle on $X$ gives rise to a height function on the set of rational points and to a new kind of adelic measures on the universal torsors over $X$.

The aim of the paper is to relate the asymptotic growth of the number of rational points of bounded height on $X$ to volumes of adelic spaces corresponding to the universal torsors over $X$.

## Introduction

One important but very difficult problem in diophantine geometry is to count the number $f(B)$ of rational points of height at most $B$ on a projective variety over a number field $k$ and to study the asymptotic growth of the counting function when $B \to \infty$. If $A \subset \mathbb{P}^n$ is an abelian variety, then it was proved by Néron (cf. [**62**]) that $f(B)/(\log B)^{\mathrm{rk}\, A(k)/2}$ converges to a constant depending on $A \subset \mathbb{P}^n$ and the height function. There is a precise adelic conjecture about the rank of $A(k)$, but this has only been established for classes of elliptic curves $E$ over $\mathbb{Q}$ for which $E(\mathbb{Q})$ is of rank 0 or 1.

For Fano varieties there is a (mostly conjectural) theory of counting functions. This theory was initiated by Manin who made some striking observations (cf. [**23**], [**3**], [**42**], [**43**]) about the counting functions for special classes of Fano varieties $X$ under their anticanonical embeddings. Similar observations for other linear systems were made by Batyrev and Manin [**3**]. The heights involved are the "usual" multiplicative heights of Weil depending on the ground field $k$ and the choice of coordinates (see [**37**, p. 50]). There may exist accumulating closed subsets with many rational points (e.g. lines on cubic surfaces). Manin therefore counts the number $f_U(B)$ of rational points of height at most $B$ on sufficiently small Zariski open $k$-subsets $U$ of $X$. He

**1991** ***Mathematics Subject Classification.*** — 14G05, 11G35, 28C99.

***Key words and phrases.*** — Fano varieties, counting functions, universal torsors, adelic measures.

notices that for a number of Fano varieties $X$ there are open $k$-subsets $U$ of $X$ such that

$$(*) \qquad f_U(B) = CB(\log B)^{\operatorname{rk}\operatorname{Pic} X - 1}(1 + o(1))$$

for an anticanonical height function. Very recently Batyrev and Tschinkel [**5**] have found that this cannot be true for all Fano varieties. But there are important classes of Fano varieties (cf. [**23**], [**52**], [**7**], [**4**]) for which asymptotic formulas of the above form have been established, and it is interesting to find common features of these results. The motivation for this paper is to obtain a better understanding of the constant $C$ in these formulas and to develop a framework which might be useful for the study of counting functions of other classes of Fano varieties.

The first systematic attempt to understand the constant $C$ in the asymptotic formulas is due to Peyre [**52**], who introduced several new ideas. He defined Tamagawa numbers for Fano varieties, thereby generalizing the classical Tamagawa numbers studied by Weil [**67**]. To define these, Peyre uses a system of $\nu$-adic "metrics" for the places $\nu$ of $k$ on the analytic anticanonical line bundle over $X(k_\nu)$ satisfying an adelic condition. He associates to any such "adelic metric" a height function $H$ on $X(k)$ and a measure on the adelic space $X(A_k)$ and suggests that the constant $C$ in $(*)$ should be equal to the product of the Tamagawa number $\tau(X)$ of $X(A_k)$ and an invariant $\alpha(X)$ depending only on effective cone in $\operatorname{Pic} X$.

Peyre assumes that the Picard group of $X_K$ for an algebraic closure $K$ of $k$ contains a $\mathbb{Z}$-basis which is invariant under the action of the Galois group $\operatorname{Gal}(K/k)$. It is clear from the work of Batyrev and Tschinkel in [**7**] and [**4**] that some restriction of this kind is needed. They prove $(*)$ for toric varieties and obtain the constant

$$(**) \qquad C = \alpha(X)\tau(X)h^1(X)$$

where $h^1(X)$ is the order of $H^1(\operatorname{Gal}(K/k), \operatorname{Pic} X_K)$.

We shall in the paper "explain" the appearance of $h^1(X)$ in $(**)$ by means of a new kind of Tamagawa numbers for universal torsors over $X$. The main idea is that the constant $C$ is related to the volumes of some adelic spaces defined by the universal torsors $\mathcal{T}$ over $X$. Universal torsors were introduced by Colliot-Thélène and Sansuc [**15**] as a generalization of the classical descent varieties of elliptic curves studied by Fermat, Mordell and Weil. The main applications of this theory so far have been in the study of the Hasse principle and weak approximation for various classes of rational varieties.

We shall use universal torsors as a natural tool when counting rational points on Fano varieties. One central idea will be to extend the height function on $X(k)$ to suitable subquotients of the adelic spaces $\mathcal{T}(A_k)$ by means of certain adelic splittings associated to the universal torsors $\pi : \mathcal{T} \to X$. For toric varieties this reduces the original counting problem to an adelic lattice point problem. We obtain thereby another proof of the asymptotic formula of Batyrev and Tschinkel [**4**] for toric varieties over $\mathbb{Q}$.

We now give a description of the content of the 11 sections of the paper.

In the first section, we study in detail the analytic manifold structure $X_{\mathrm{an}}(k_\nu)$ on $X(k_\nu)$ and metrics (which we will call *norms* from now on) on analytic line bundles on $X_{\mathrm{an}}(k_\nu)$. For anticanonical line bundles with a norm we recall the measure constructed by Peyre and explain its relation to the classical construction of a measure from a global differential form. For submersions

$$Y_{\mathrm{an}}(k_\nu) \longrightarrow X_{\mathrm{an}}(k_\nu)$$

we give a relative version of Peyre's construction and associate a positive linear map

$$\Lambda : C_c(Y_{\mathrm{an}}(k_\nu)) \longrightarrow C_c(X_{\mathrm{an}}(k_\nu))$$

to norms on relative anticanonical line bundles.

In section 2, we assume that $\nu$ is non-archimedean and study norms for the compact open subsets $\Xi_\nu(o_\nu) \subseteq X_{\mathrm{an}}(k_\nu)$ defined by models $\Xi_\nu$ of $X \times k_\nu$ over the valuation ring $o_\nu$ in $k_\nu$. We relate in (2.14) the volume of $\Xi_\nu(o_\nu)$ with respect to the measure determined by the above norm to the density of the reduction of $\Xi_\nu(o_\nu)$ modulo finite powers of the maximal ideal $m_\nu$ in $o_\nu$. As a consequence, we get an explicit formula (cf. (2.14)(b)) for the volume of $\Xi_\nu(o_\nu)$ with respect to any measure defined by a norm on $\Xi_\nu(o_\nu)$. This formula holds also when $\Xi_\nu$ has bad reduction and $X$ is non proper. If $X$ is proper, then $\Xi_\nu(o_\nu) = X_{\mathrm{an}}(k_\nu)$ and we get a formula for the volume of $X_{\mathrm{an}}(k_\nu)$. But there are also important applications of this formula to universal torsors and other non-proper varieties.

In section 3, we study invariant norms over local fields on the relative anticanonical line bundles for $X$-torsors $\pi : \mathcal{T} \to X$ under arbitrary algebraic groups $G$. We then concentrate on the norms defined by relative differential forms. If $G$ is a torus $T$, then there is a canonical norm of this kind which we will baptize the *order norm*. Now using this relative norm we obtain for each norm on $X_{\mathrm{an}}(k_\nu)$ an "induced" norm on $\mathcal{T}_{\mathrm{an}}(k_\nu)$ which in its turn defines an "induced" measure on $\mathcal{T}_{\mathrm{an}}(k_\nu)$.

In section 4, we consider varieties $X$ over number fields $k$ and the adelic topological space $X(A_k)$. We have not found any modern rigorous version of Weil's account **[66]** and we therefore explain how to use schemes of finite presentation in EGA to develop the foundations for adelic spaces. We then generalize Peyre's notion of "adelic metric" and his adelic measures in many ways. We introduce e.g. relative adelic norms for smooth morphisms $\pi : Y \to X$ over $X$ and positive linear maps $\Lambda : C_c(Y(A_k)) \to C_c(X(A_k))$. It is thereby necessary to consider convergence factors which vary among the fibres and to consider fibres over $A_k$ not defined over $k$ even if $\pi : Y \to X$ is defined over $k$.

In section 5, we restrict to torsors $p : \mathcal{T} \to X$ for varieties over number fields and study adelic norms and measures for them. When $X$ is smooth and proper with

$$H^1_{\mathrm{Zar}}(X, \mathcal{O}_X) = H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$$

and with torsion-free Néron-Severi group, then there is a notion of universal torsors $\pi : \mathcal{T} \to X$. The $\nu$-adic order norms form an adelic norm which in its turn gives rise to a positive linear map

$$\Lambda : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k)).$$

This map depends on the choice of convergence factors. But one can choose these to be inverse to the convergence factors for $X(A_k)$ so that the induced measure on $C_c(\mathcal{T}(A_k))$ requires no convergence factors. Using this measure on $\mathcal{T}(A_k)$, we define Tamagawa numbers for universal torsors.

In section 6, we recall the Brauer group and torsor obstructions to weak approximation of Manin, Colliot-Thélène and Sansuc. By using their theory together with results of Ono on the arithmetic of tori, we relate our Tamagawa numbers for universal torsors to the Tamagawa numbers of Peyre. The factor $h^1(X)$ enters naturally.

In section 7, we modify the original growth conjectures of Manin in order to exclude the Fano varieties containing infinitely many weakly accumulating subvarieties. This is not so original and closely related to notions of Manin and Peyre (cf. **[51]**). We also generalize and refine Peyre's Tamagawa conjecture for Fano varieties by means of our Tamagawa numbers for universal torsors.

In section 8, we study the geometry of universal torsors $\pi : \mathcal{T} \to X$ over smooth complete toric varieties which are trivial over the unit element of the $k$-torus $U$ in $X$. We identify them with the toric varieties studied by Cox in his article **[16]** and find that they are open subsets of affine spaces.

In section 9, we consider toric varieties over local fields $k_\nu$. We give an explicit description of the norms for universal torsors obtained by inducing the norms of Batyrev-Tschinkel **[7]** and of the corresponding measures (cf. (9.12)). Another central idea is the introduction of a *canonical toric splitting*

$$\check{\psi}_\nu : X(k_\nu) \longrightarrow \mathcal{T}(k_\nu)/T(k_\nu)_{\text{cp}}$$

of the map

$$\check{\pi}_\nu : \mathcal{T}(k_\nu)/T(k_\nu)_{\text{cp}} \longrightarrow X(k_\nu).$$

induced by the principal universal torsor $\pi : \mathcal{T} \to X$. Here $T(k_\nu)_{\text{cp}}$ is the maximal compact subgroup of the analytic group $T(k_\nu)$ defined by the Néron-Severi torus.

In section 10, we consider toric varieties over number fields and the induced adelic norm on the universal torsor obtained from the induced $\nu$-adic norms in section 9. The product map of all $\check{\psi}_\nu$ gives rise to a continuous canonical toric splitting

$$\check{\psi}_A : X(A_k) \longrightarrow \mathcal{T}(A_k)/T(A_k)_{\text{cp}}$$

of the map from $\mathcal{T}(A_k)/T(A_k)_{\text{cp}}$ to $X(A_k)$ induced by $\pi$. By means of $\check{\psi}$ we give a new torsor theoretic interpretation of the heights of Batyrev-Tschinkel **[7]** and an interpretation of the constant $C = \alpha(X)\tau(X)h^1(X)$. There are several analogies with Bloch's use of torsors (cf. **[8]**, **[47]**) to interpret the Néron-heights and the Birch/Swinnerton-Dyer/Tate conjecture for abelian varieties. The universal torsors