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CHARACTERIZATION OF THE RADON-NIKODYM PROPERTY IN TERMS OF INVERSE LIMITS

by

Jeff Cheeger & Bruce Kleiner

Abstract. — In this paper we clarify the relation between inverse systems, the Radon-Nikodym property, the Asymptotic Norming Property of James-Ho [**10**], and the GFDA spaces introduced in [**5**].

***Résumé* (Caractérisation de la propriété de Radon-Nikodym en termes de limites inverses)**
Dans cet article nous clarifions la relation entre les systèmes inverses, la propriété de Radon-Nikodym, la propriété normative asymptotique de James-Ho [**10**] et les espaces GFDA, introduits dans [**5**].

1. Introduction

A Banach space V is said to have the *Radon-Nikodym Property* (RNP) if every Lipschitz map $f : \mathbf{R} \to V$ is differentiable almost everywhere. By now, there are a number of characterizations of Banach spaces with the RNP, the study of which goes back to Gelfand [**7**]; for additional references and discussion, see [**1**, Chapter 5], [**8**]. Of particular interest here is the characterization of the RNP in terms of the Asymptotic Norming Property; [**10, 8**].

In this paper we will show that a variant of the GFDA property introduced in [**5**] is actually equivalent to the Asymptotic Norming property of James-Ho, and hence by [**10, 8**], is equivalent to the RNP. In addition, we observe that the GFDA spaces of [**5**] are just spaces which are isomorphic to a separable dual space.

Definition 1.1. — An inverse system

$$W_1 \xleftarrow{\theta_1} W_2 \xleftarrow{\theta_2} \dots \xleftarrow{\theta_{i-1}} W_i \xleftarrow{\theta_i} \dots, \tag{1.2}$$

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is *standard* if the W_i's are finite dimensional Banach spaces and the θ_i's are linear maps of norm ≤ 1. We let $\pi_j : \underleftarrow{\lim}\, W_i \to W_j$ denote the projection map.

Definition 1.3. — Let $\{(W_i, \theta_i)\}$ be a standard inverse system and $V \subset \underleftarrow{\lim}\, W_i$ be a subspace. The pair $(\underleftarrow{\lim}\, W_i, V)$ has the *Determining Property* if a sequence $\{v_k\} \subset V$ converges strongly provided the projected sequences $\{\pi_j(v_k)\} \subset W_j$ converge for every j, the sequence $\{\|v_k\|\}$ is bounded, and the convergence $\|\pi_j(v_k)\| \to \|v_k\|$ is uniform in k. A Banach space U has the *Determining Property* if there is a pair $(\underleftarrow{\lim}\, W_i, V)$ with Determining Property, such that V is isomorphic to U.

We have:

Theorem 1.4. — *A separable Banach space has the RNP if and only it has the Determining Property.*

Since a Banach space has the RNP if and only if every separable subspace has the RNP, Theorem 1.4 yields a characterization of the RNP for nonseparable Banach spaces as well.

To prove the theorem, we first observe in Proposition 2.8 that the inverse limit $\underleftarrow{\lim}\, W_i$ is the dual space of a separable Banach space. Then, by a completely elementary argument, we show that a Banach space has the Determining Property if and only if it has the Asymptotic Norming Property (ANP) of James-Ho [**10**]. Since a separable Banach space U has the RNP if and only if it has the ANP [**10, 8**], the theorem follows. We remark that there is a simple direct proof that if V has the ANP (or the Determining Property), then every Lipschitz map $f : \mathbf{R} \to V$ is differentiable almost everywhere.

Characterizations of the RNP using inverse limits are useful for applications; see [**5**], the discussion below concerning metric measure spaces, and [**6**].

Relation with previous work. — In slightly different language, our earlier paper [**5**] also considered pairs $(\underleftarrow{\lim}\, W_i, V)$, where $\underleftarrow{\lim}\, W_i$ is the inverse limit of a standard inverse system, and $V \subset \underleftarrow{\lim}\, W_i$ is a closed subspace. A Good Finite Dimensional Approximation (GFDA) of a Banach space V, a notion introduced in [**5**], is a pair $(\underleftarrow{\lim}\, W_i, V)$ with the Determining Property such that $\pi_i\big|_V : V \to W_i$ is a quotient map for every i.

It follows immediately from Lemma 3.8 of [**5**] that if $(\underleftarrow{\lim}\, W_i, V)$ is a GFDA of V, then $V = \underleftarrow{\lim}\, W_i$. Since such inverse limits are dual spaces by Proposition 2.8, V is a separable dual space in this case. Conversely, using the Kadec-Klee renorming Lemma [**11, 12**], it was shown in [**5**] that every separable dual space is isomorphic to a Banach space which admits a GFDA. Thus, a Banach space admits a GFDA if and only if it is isomorphic to a separable dual space.

Applications to metric measure spaces. — We will call a metric measure space (X, μ) a *PI space* if the measure is doubling, and a Poincaré inequality holds in the sense of upper gradients [**9, 4**]. In [**5**], differentiation and bi-Lipschitz non-embedding theorems were proved for maps $f : X \to V$ from PI spaces into GFDA targets V, generalizing results of [**4**] for finite dimensional targets. As explained above, it turns out that these targets are just separable dual spaces, up to isomorphism.

As an application of the inverse limit framework, we will show in [**6**] that the differentiation theorem [**5**, Theorem 4.1] and bi-Lipschitz non-embedding theorem [**5**, Theorem 5.1] hold whenever the target has the RNP.

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2. Inverse systems

In this section, we recall some basic facts concerning direct and inverse systems, and the duality between them. Then we show that inverse limits of standard inverse systems are precisely duals of separable spaces.

The following conventions will be in force throughout the remainder of the paper.

Definition 2.1. — An *standard direct system* is a sequence of finite dimensional Banach spaces $\{E_i\}$ and 1-Lipschitz linear maps $\iota_i : E_i \to E_{i+1}$.

Definition 2.2. — An *standard inverse system* is a sequence of finite dimensional Banach spaces $\{W_i\}$ and 1-Lipschitz linear maps $\theta_i : W_{i+1} \to W_i$.

Definition 2.3. — A standard direct system is *isometrically injective* if the maps $\iota_i : E_i \to E_{i+1}$ are isometric injections.

Definition 2.4. — A standard inverse system is *quotient* if the maps $\theta_i : W_{i+1} \to W_i$ are quotient maps.

By a *quotient map* of normed spaces, we mean a surjective map $\pi : U \to V$ for which the norm on the target is the quotient norm, i.e. for every $v \in V$,

$$\|v\| = \inf\{ \|u\| \mid u \in \pi^{-1}(v)\}.$$

We will refer to the maps ι_i and θ_i as *bonding maps.*

There is a duality between the objects in Definitions 2.1 and 2.2, respectively, 2.3 and 2.4: if $\{(E_i, \iota_i)\}$ is a standard direct system, then $\{(E_i^*, \iota_i^*)\}$ is a standard inverse system and conversely; similarly, isometrically injective direct systems are dual to quotient systems. To see this, one uses the facts that the adjoint of a 1-Lipschitz map of Banach spaces is 1-Lipschitz and the the adjoint of an isometric embedding is a quotient map. (This follows from the Hahn-Banach theorem.) In particular, since the spaces in our systems are assumed to be finite dimensional (hence reflexive) every inverse system arises as the dual of its dual direct system and conversely. The same holds for quotient inverse systems.

We now recall the definitions of direct and inverse limits.

Given a standard direct system $\{(E_i, \iota_i)\}$ we form the direct limit Banach space $\varinjlim E_i$ as follows. We begin with the disjoint union $\sqcup_i E_i$, and declare two elements $e \in E_i$, $e' \in E_{i'}$ to be equivalent if their images in E_j coincide for some $j \geq \max\{i, i'\}$. Since the bonding maps are 1-Lipschitz, the set of equivalence classes inherits an obvious vector space structure with a pseudo-norm. The direct limit $\varinjlim E_i$ is defined to be the completion of the quotient of this space by the closed subspace of elements whose pseudo-norm is zero. Clearly, there are 1-Lipschitz maps

$$\tau_i : E_i \to \varinjlim E_i\,,$$

which in the case of isometrically injective direct systems, are isometric injections. The union $\bigcup_i \tau_i(E_i)$ is dense in $\varinjlim E_i$.

The inverse limit $\varprojlim W_i$ of a standard inverse system $\{(W_i, \theta_i)\}$ is defined as follows. The underlying set consists of the collection of elements $(w_i) \in \prod_i W_i$ which are compatible with the bonding maps, i.e. $\theta_i(w_i) = w_{i-1}$ for all i, and which satisfy $\sup_i \|w_i\| < \infty$. This is equipped with the obvious vector space structure and the norm

$$\|\{w_i\}\| := \lim_{j\to\infty} \|w_j\|\,. \tag{2.5}$$

The map

$$\pi_j : \varprojlim W_i \to W_j \tag{2.6}$$

given by

$$\pi_j(\{w_i\}) = w_j$$

is 1-Lipschitz, and

$$\lim_{j\to\infty} \|\pi_j(\{w_i\})\| = \|\{w_i\}\|\,.$$

An inverse limit $\varprojlim W_i$ has a natural *inverse limit topology*, namely the weakest topology such that every projection map $\pi_j : \varprojlim W_i \to W_j$ is continuous. Thus a sequence $\{v_k\} \subset \varprojlim W_i$ converges in the inverse limit topology to $v \in \varprojlim W_i$ if and only if for every i, we have $\pi_i(v_k) \to \pi_i(v)$ as $k \to \infty$.

If $\{v_k\} \subset \varprojlim W_i$ and $\{v_k\} \overset{\text{invlim}}{\longrightarrow} v \in \varprojlim W_i$, then

$$\|v\| \leq \liminf_k \|v_k\|\,. \tag{2.7}$$

Also, every norm bounded sequence $\{v_k\} \subset \varprojlim W_i$ has a subsequence which converges with respect to the inverse limit topology; this follows from a diagonal argument, because $\{\pi_i(v_k)\}$ is contained in a compact subset of W_i, for all i.