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MANIN'S AND PEYRE'S CONJECTURES ON RATIONAL POINTS AND ADELIC MIXING*

BY Alex GORODNIK, François MAUCOURANT AND Hee OH

Dedicated to Prof. Gregory Margulis on the occasion of his sixtieth birthday

ABSTRACT. – Let X be the wonderful compactification of a connected adjoint semisimple group G defined over a number field K. We prove Manin's conjecture on the asymptotic (as $T \to \infty$) of the number of K-rational points of X of height less than T, and give an explicit construction of a measure on $X(\mathbb{A})$, generalizing Peyre's measure, which describes the asymptotic distribution of the rational points $\mathbf{G}(K)$ on $X(\mathbb{A})$. Our approach is based on the mixing property of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ which we obtain with a rate of convergence.

RÉSUMÉ. – Soit X la compactification merveilleuse d'un groupe semi-simple $\mathbf{G}$, connexe, de type adjoint, algébrique défini sur un corps de nombre K. Nous démontrons l'asymptotique conjecturée par Manin du nombre de points K-rationnels sur X de hauteur plus petite que T, lorsque $T \to +\infty$, et construisons de manière explicite une mesure sur $X(\mathbb{A})$, généralisant celle de Peyre, qui décrit la répartition asymptotique des points rationnels $\mathbf{G}(K)$ sur $X(\mathbb{A})$. Ce travail repose sur la propriété de mélange de $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, qui est démontrée avec une estimée de vitesse.

1. Introduction

Let K be a number field and X a smooth projective variety defined over K. A fundamental problem in modern algebro-arithmetic geometry is to describe the set $X(K)$ in terms of the geometric invariants of X. One of the main conjectures in this area was made by Manin in the late eighties in [1]. It formulates the asymptotic (as $T \to \infty$) of the number of points in $X(K)$ of height less than T for Fano varieties (that is, varieties with ample anti-canonical class).

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Manin's conjecture has been proved for flag varieties ([26], [41]), toric varieties ([3], [2]), horospherical varieties [52], equivariant compactifications of unipotent groups (see [13], [49], [47]), etc. We refer to survey papers by Tschinkel ([56], [57]) for a more precise background on this conjecture. Recently Shalika, Tschinkel and Takloo-Bighash proved the conjecture for the wonderful compactification of a connected semisimple adjoint group [48]. In this paper, we present a different proof of the conjecture, as well as describe the asymptotic distributions of rational points of bounded height as conjectured by Peyre. Our proof relies on the computation of the volume asymptotics of height balls in [48]. We refer to [53] for the comparison of these two approaches.

Although our work is highly motivated by conjectures in arithmetic geometry, our approach is almost purely (algebraic) group theoretic. For this reason, we formulate our main results in the language of algebraic groups and their representations in the introduction and refer to section 7 for the account that how these results imply the conjectures of Manin and Peyre.

1.1. Height function

We begin by defining the notion of a height function on the K-rational points of the projective n-space $\mathbb{P}^n$. Intuitively speaking, the height of a rational point $x \in \mathbb{P}^n(K)$ measures an *arithmetic* size of x. In the case of $K = \mathbb{Q}$, it is simply given by

$$\mathrm{H}(x) = \max_{0 \le i \le n} |x_i|$$

where $(x_0, \ldots, x_n)$ is a primitive integral vector representing x. To give its definition for a general K, we denote by R the set of all normalized absolute values $x \mapsto |x|_v$ of K, and by K_v the completion of K with respect to $|\cdot|_v$. For each $v \in R$, choose a norm H_v on K_v^{n+1} which is simply the max norm $\mathrm{H}_v(x_0, \ldots, x_n) = \max_{i=0}^n |x_i|_v$ for almost all v. Then a function $\mathrm{H} : \mathbb{P}^n(K) \to \mathbb{R}_{>0}$ of the following form is called a height function:

$$\mathrm{H}(x) := \prod_{v \in R} \mathrm{H}_v(x_0, \ldots, x_n)$$

for $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(K)$. Since $\mathrm{H}_v(x_0, \ldots, x_n) = 1$ for almost all $v \in R$, we have $\mathrm{H}(x) > 0$ and by the product formula, H is well defined, i.e., independent of the choice of representative for x.

It is easy to see that for any $T > 0$, the number

$$N(T) := \#\{x \in \mathbb{P}^n(K) : \mathrm{H}(x) < T\}$$

is finite. Schanuel [45] computed the precise asymptotic in 1964:

$$N(T) \sim c \cdot T^{n+1} \quad \text{as } T \to \infty$$

for some explicit constant $c = c(\mathrm{H}) > 0$.

Unless mentioned otherwise, throughout the introduction, we let $\mathbf{G}$ be a connected semisimple adjoint group over K and $\iota : \mathbf{G} \to \mathrm{GL}_N$ be a faithful representation of $\mathbf{G}$ defined over K with a unique maximal weight. Consider the projective embedding of $\mathbf{G}$ over K induced by ι:

$$\bar{\iota} : \mathbf{G} \to \mathbb{P}(\mathrm{M}_N)$$

where M_N denotes the space of matrices of order N. We then define a height function H_ι on $\mathbf{G}(K)$ associated to ι by pulling back a height function on $\mathbb{P}(\mathrm{M}_N(K))$ via $\bar{\iota}$. That is, for $g \in \mathbf{G}(K)$,

$$\mathrm{H}_\iota(g) := \prod_{v \in R} \mathrm{H}_v(\iota(g)), \tag{1.1}$$

where each H_v is a norm on $\mathrm{M}_N(K_v)$, which is the max norm for almost all $v \in R$.

We note that H_ι is not uniquely determined by ι, because of the freedom of choosing H_v locally (though only for finitely many v).

1.2. Asymptotic number of rational points

For each $T > 0$, we introduce the notation for the number of points in $\mathbf{G}(K)$ of height less than T:

$$N(\mathrm{H}_\iota, T) := \#\{g \in \mathbf{G}(K) \ : \ \mathrm{H}_\iota(g) \ < \ T\}.$$

Theorem 1.2. – *There exist $a_\iota \in \mathbb{Q}^+$, $b_\iota \in \mathbb{N}$ and $c = c(\mathrm{H}_\iota) > 0$ such that for some $\delta > 0$,*

$$N(\mathrm{H}_\iota, T) = c \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \cdot (1 + O((\log T)^{-\delta})).$$

The constants a_ι and b_ι can be defined explicitly by combinatorial data on the root system of $\mathbf{G}$ and the unique maximal weight of ι. Choose a maximal torus $\mathbf{T}$ of $\mathbf{G}$ defined over K containing a maximal K-split torus and a set Δ of simple roots in the root system $\Phi(\mathbf{G}, \mathbf{T})$. Denote by 2ρ the sum of all positive roots in $\Phi(\mathbf{G}, \mathbf{T})$, and by λ_ι the maximal weight of ι. Define $u_\alpha, m_\alpha \in \mathbb{N}$, $\alpha \in \Delta$, by

$$2\rho = \sum_{\alpha \in \Delta} u_\alpha \alpha \quad \text{and} \quad \lambda_\iota = \sum_{\alpha \in \Delta} m_\alpha \alpha.$$

The fact that $m_\alpha \in \mathbb{N}$ follows since $\mathbf{G}$ is of adjoint type. Consider the twisted action of the Galois group $\Gamma_K := \mathrm{Gal}(\bar{K}/K)$ on Δ (for instance, if the K-form of $\mathbf{G}$ is inner, this action is just trivial). Then

$$a_\iota = \max_{\alpha \in \Delta} \frac{u_\alpha + 1}{m_\alpha} \quad \text{and} \quad b_\iota = \# \left\{ \Gamma_K . \alpha : \frac{u_\alpha + 1}{m_\alpha} = a_\iota \right\}. \tag{1.3}$$

Note that the exponent a_ι is independent of the field K, and b_ι depends only on the quasi-split K-form of $\mathbf{G}$. Therefore, by passing to a finite field extension containing the splitting field of $\mathbf{G}$, b_ι also becomes independent of K.

Remark 1.1. – When $\mathbf{G}$ is almost K-simple or, more generally, when H_ι is the product of height functions of the K-simple factors of $\mathbf{G}$, we can improve the rate of convergence in Theorem 1.2: for some $\delta > 0$,

$$N(\mathrm{H}_\iota, T) = c \cdot T^{a_\iota} P(\log T) \cdot \left(1 + O\left(T^{-\delta}\right)\right)$$

where $P(x)$ is a monic polynomial of degree $b_\iota - 1$.

1.3. Distribution of rational points

For each $v \in R$, denote by $X_{\iota,v}$ the closure of $\bar{\iota}(\mathbf{G}(K_v))$ in $\mathbb{P}(\mathrm{M}_N(K_v))$, and consider the compact space $X_\iota := \prod_{v\in R} X_{\iota,v}$. In section 6, we construct a probability measure μ_ι on X_ι which describes the asymptotic distribution of rational points in $\mathbf{G}(K)$ in X_ι with respect to the height H_ι. To keep the introduction concise, we give the definition of μ_ι only when ι is saturated. A representation $\iota : \mathbf{G} \to \mathrm{GL}_N$ is called *saturated* if the set

$$\left\{\alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\right\}$$

is not contained in the root system of a proper normal K-subgroup of $\mathbf{G}$. In particular, if $\mathbf{G}$ is almost K-simple, any representation of $\mathbf{G}$ is saturated.

Let τ denote the Haar measure on $\mathbf{G}(\mathbb{A})$ such that $\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = 1$. Denote by Λ the set of all automorphic characters of $\mathbf{G}(\mathbb{A})$ (cf. section 2.4) and by W_ι the maximal compact subgroup of the group $\mathbf{G}(\mathbb{A}_f)$ of finite adeles, under which H_ι is bi-invariant (see Definition 2.7). Then the following is a positive real number (see Propositions 4.6 and 4.11 (3), noting $r_\iota = \gamma_{W_\iota}(e)$ in the notation therein):

$$r_\iota := \sum_{\chi\in\Lambda} \lim_{s\to a_\iota^+} (s - a_\iota)^{b_\iota} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g)^{-s}\chi(g)\, d\tau(g). \tag{1.4}$$

For ι saturated, the probability measure μ_ι on X_ι is the unique measure satisfying that for any $\psi \in C(X_\iota)$ invariant under a co-finite subgroup of W_ι,

$$\mu_\iota(\psi) = r_\iota^{-1} \cdot \sum_{\chi\in\Lambda} \lim_{s\to a_\iota^+} (s - a_\iota)^{b_\iota} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g)^{-s}\chi(g)\, \psi(g)\, d\tau(g) \tag{1.5}$$

(see Theorem 4.18). We refer to (6.16) for the definition of μ_ι for a general ι:

THEOREM 1.6. – *For any $\psi \in C(X_\iota)$,*

$$\lim_{T\to\infty} \frac{1}{N(\mathrm{H}_\iota, T)} \sum_{g\in\mathbf{G}(K)\,:\,\mathrm{H}_\iota(g)<T} \psi(g) = \int_{X_\iota} \psi\, d\mu_\iota.$$

REMARK 1.7. –
1. For ι saturated, the measure μ_ι coincides with the measure $\tilde{\mu}_\iota$ which describes the distribution of height balls in $\mathbf{G}(\mathbb{A})$ (see Proposition 4.27).
2. Although the projection $\mu_{\iota,S}$ of μ_ι to $X_{\iota,S} = \prod_{v\in S} X_{\iota,v}$ is always equivalent to a Haar measure on $\mathbf{G}_S = \prod_{v\in S} \mathbf{G}(K_v)$ (Proposition 4.22), it is $\mathbf{G}_S$-invariant, only when the height $\mathrm{H}_{\iota,S} = \prod_{v\in S} \mathrm{H}_v \circ \iota$ is $\mathbf{G}_S$-invariant.
3. The space $X_{\iota,S}$ is a compactification of $\mathbf{G}_S$ which is an analog of the Satake compactification defined for real groups (see, for example, [7]). Theorem 1.6 implies that the rational points $\mathbf{G}(K)$ do not escape to the boundary $X_{\iota,S} - \mathbf{G}_S$. It is interesting to compare this result with the distribution of the integral points $\mathbf{G}(\mathbb{Z})$ of bounded height in the Satake compactification of $\mathbf{G}(\mathbb{R})$ where the limiting distribution is supported on the boundary (see [29] and [38] for more details).