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CROSS RATIOS, ANOSOV REPRESENTATIONS AND THE ENERGY FUNCTIONAL ON TEICHMÜLLER SPACE

BY François LABOURIE

ABSTRACT. – We study two classes of linear representations of a surface group: Hitchin and maximal symplectic representations. We relate them to cross ratios and thus deduce that they are *displacing* which means that their translation lengths are roughly controlled by the translations lengths on the Cayley graph. As a consequence, we show that the mapping class group acts properly on the space of representations and that the energy functional associated to such a representation is proper. This implies the existence of minimal surfaces in the quotient of the associated symmetric spaces, a fact which leads to two consequences: a rigidity result for maximal symplectic representations and a partial result concerning a purely holomorphic description of the Hichin component.

RÉSUMÉ. – Nous étudions deux classes de représentations linéaires d'un groupe de surface : les représentations de Hitchin et les représentations symplectiques maximales. En reliant ces représentations à des birapports, nous montrons qu'elles sont *déplaçantes*, c'est-à-dire que leurs longueurs de translation sont grossièrement contrôlées par celles du graphe de Cayley. Ceci nous permet de montrer que le groupe modulaire agit proprement sur l'espace de ces représentations et que la fonctionnelle énergie associée à une telle représentation est propre. Nous en déduisons alors l'existence de surfaces minimales dans les quotients d'espaces symétriques associés et en tirons deux conséquences : un résultat de rigidité pour les représentations symplectiques et un résultat partiel concernant la description de la composante de Hitchin en termes purement holomorphes.

1. Introduction

Let S be a closed connected oriented surface of genus greater than one. Monodromies of hyperbolic structures on S define a distinguished class of homomorphisms from the fundamental group $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$. In this paper we study two generalisations of these surface group representations, one in which we replace $\mathrm{PSL}(2,\mathbb{R})$ by $\mathrm{PSL}(n,\mathbb{R})$ and one in which we generalise $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{PSp}(2,\mathbb{R})$ to $\mathrm{PSp}(2n,\mathbb{R})$.

The first generalisation uses the irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{PSL}(n,\mathbb{R})$. A *Fuchsian representation* from $\pi_1(S)$ into $\mathrm{PSL}(n,\mathbb{R})$ is a representation which decomposes as the product of a faithful cocompact representation from $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$ and the irreducible representation from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{PSL}(n,\mathbb{R})$ (see [30] and Section 4.1). The representations we study, called *Hitchin representations*, are those which may be *deformed* into a Fuchsian representation. In [26], Hitchin studies the moduli space of reductive (*i.e.* whose Zariski closure is a reductive group) Hitchin representations. For a combinatorial point of view on a related subject, see Fock and Goncharov in [16].

The second generalisation exploits the fact that the homogeneous space M associated to $\mathrm{PSp}(2n,\mathbb{R})$ is Hermitian symmetric and thus carries an invariant symplectic form ω (see Section 4.2.1 for details). Given a representation ρ from $\pi_1(S)$ to $\mathrm{PSp}(2n,\mathbb{R})$, if f is any ρ-equivariant map from the universal cover of S to M, then $f^*\omega$ is invariant under the action of $\pi_1(S)$. The following number

$$\tau(\rho)=\frac{n}{2\pi}\int_S f^*\omega$$

is then an integer independent of the choice of f. This number, called the *Toledo invariant* of ρ, remains constant under continuous deformations of the representation and satisfies a generalised Milnor-Wood Inequality (see [43])

$$|\tau(\rho)|\leqslant n|\chi(S)|.$$

By definition, a *maximal symplectic representation* is one for which the Toledo invariant attains the upper bound in this inequality. The notion of maximality and a suitable version of the Milnor-Wood Inequality extend to all Hermitian symmetric spaces. W. Goldman shows in [18, 19] that maximal representations in $\mathrm{PSL}(2,\mathbb{R})$ are precisely monodromies of hyperbolic structures. In the general case, these maximal representations have been extensively studied by Bradlow, García-Prada, Gothen, Mundet i Riera (as well as Xia in a specific example) ([3, 5, 4, 17, 22, 45]) using Higgs bundle techniques on one hand and Burger, Iozzi and Wienhard ([6, 9, 8, 44]) using bounded cohomology techniques on the other hand.

Both type of representations – Hitchin representations and maximal symplectic representations – can be thought of as generalisations of the $\mathrm{PSL}(2,\mathbb{R})$-representations which arise from monodromies of hyperbolic structures and hence as generalising Teichmüller-Thurston theory.

The maximal symplectic representations and the Hitchin representations are known to share several fundamental properties, including:

- They are Anosov as defined in [30]. For Hitchin representations this is proved in [30], for maximal representations this is shown in [7] by Burger, Iozzi, Wienhard and the author.
- The Zariski closure of the images are reductive. For Hitchin representations, see Proposition 4.1.5. For maximal representations, this is proved by Burger, Iozzi and Wienhard in [8].
- They are discrete: see [30] for Hitchin representations and the proof by Burger, Iozzi and Wienhard in [8] for maximal representations.

The results presented in this paper extend this list of common features by showing that both types of representation have the property that we call *displacing*. More precisely, let Γ be a finitely generated subgroup of the isometry group of a metric space X: we say that Γ is *displacing* if, given a finite generating set $\mathcal{G}$ of Γ, there exist positive constants A and B such that for all elements γ of Γ

$$\inf_{x\in X} d(x,\gamma(x)) \geqslant A \inf_{\eta\in\Gamma} \|\eta\gamma\eta^{-1}\|_{\mathcal{G}} - B,$$

where $\|\gamma\|_{\mathcal{G}}$ is the word length of γ with respect to $\mathcal{G}$. It is easy to check that this definition is independent of the generating set $\mathcal{G}$. Note that cocompact groups are always displacing, as are convex-cocompact groups whenever X is *Hadamard* (*i.e.* complete, nonpositively curved and simply connected). If ρ is a representation from a finitely generated group Γ with values in a connected semi-simple real Lie group G without compact factor and with trivial centre, then ρ is *displacing* if the group $\rho(\Gamma)$ is displacing as a group of isometries of the associated symmetric space.

We now briefly summarise results of Delzant, Guichard, Mozes and the author in [13] which compare this notion to the fact that orbit maps are quasiisometries. While the two notions turn out to be equivalent for surface groups and more generally hyperbolic groups, they are not equivalent for every group: there are known examples which have displacing representations whose orbit maps are not quasiisometries and also nondisplacing representations for which the orbit maps are quasiisometries.

The starting point of this article is the following result.

THEOREM 1.0.1. – *Hitchin and maximal symplectic representations are displacing.*

It has already been observed by Burger, Iozzi, Wienhard and the author in [7] that the orbit maps are quasiisometries for maximal symplectic representations. Here we prove the theorem by relating Hitchin representations and maximal symplectic representations to cross ratios (cf. Theorems 4.1.6 and 4.2.4).

The two main applications of this result are that

– the mapping class group acts properly on certain moduli spaces, and
– the energy functional on Teichmüller space is proper.

Let us be more specific. Let $\mathrm{Hom}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$ be the space of Hitchin homomorphisms and

$$\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})) = \mathrm{Hom}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R}),$$

where the action of $\mathrm{PSL}(n,\mathbb{R})$ is by conjugation.

Similarly, let $\mathrm{Hom}_T(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$ be the space of maximal symplectic homomorphisms and

$$\mathrm{Rep}_T(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R})) = \mathrm{Hom}_T(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))/\mathrm{PSp}(2n,\mathbb{R}).$$

Both spaces Hom_H and Hom_T are unions of connected components, each of which is a component of the corresponding space of reductive homomorphisms. Moreover, since they consist of reductive representations, the quotient spaces Rep_H and Rep_T are Hausdorff (hence locally compact). This last fact follows from the identification due to Hitchin [25] of reductive representations with polystable Higgs bundle, although more direct proofs could be obtained. The *mapping class group* $\mathcal{M}(S)$ – that is the group of outer automorphisms of $\pi_1(S)$ which may be represented by orientation preserving diffeomorphisms – acts by precomposition on these spaces. The following result will be an immediate consequence of Theorem 1.0.1.

THEOREM 1.0.2. – *The mapping class group* $\mathcal{M}(S)$ *acts properly by precomposition on the spaces* $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$ *and* $\mathrm{Rep}_T(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$.

In the case of $\mathrm{PSL}(3,\mathbb{R})$, W. Goldman proved in [20] that the mapping class group acts properly on the moduli space of convex $\mathbb{RP}^2$ structures. Moreover, together with Choi in [10], he identified this moduli space with the Hitchin component.

Our second main application concerns the energy functional. We first recall briefly the general framework and refer to Paragraph 5 for precise definitions. Let ρ be a representation from $\pi_1(S)$ to a connected semi-simple real Lie group G without compact factor and with trivial centre. Let M be the symmetric space associated to G and let M_ρ be the flat M-bundle over S defined by the representation ρ. Let $\Gamma(S, M_\rho)$ be the space of smooth sections of M_ρ. If J is a complex structure on S and f an element of $\Gamma(S, M_\rho)$, we define

$$\mathrm{Energy}_J(f) = \int_S \langle df \wedge df \circ J \rangle.$$

Then, the *energy functional* e_ρ, associated to the representation ρ, is the map from the space of all complex structures on S to the real numbers defined by

$$e_\rho(J) = \inf\{\mathrm{Energy}_J(f) \mid f \in \Gamma(S, M_\rho)\}.$$

The value of this function depends only on the isotopy class of the complex structure J, and hence defines a function on Teichmüller space. Denoted by e_ρ and also called the *energy functional*, this function is smooth on Teichmüller space (cf. Paragraph 5.2). We shall prove the following result

THEOREM 1.0.3. – *If* ρ *is a Hitchin representation or a maximal symplectic representation, then the energy functional* e_ρ *is a proper function on Teichmüller space.*

It is classical that critical points of the energy functional are related to minimal surfaces. Indeed, using Gulliver's definition of a branched immersion ([24]), we obtain the following consequence

COROLLARY 1.0.4. – *Let* ρ *be a Hitchin or maximal symplectic representation. Then there exists a minimal branched immersion from* S *into* $M/\rho(\pi_1(S))$ *which represents* ρ *at the level of homotopy groups.*

This corollary will lead to two applications that we shall explain in the next section.

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