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Paolo CASCINI & Hiromu TANAKA & Chenyang XU

*On base point freeness in positive characteristic*

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# ON BASE POINT FREENESS IN POSITIVE CHARACTERISTIC

BY PAOLO CASCINI, HIROMU TANAKA AND CHENYANG XU

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**ABSTRACT.** – We prove that if  $(X, A + B)$  is a pair defined over an algebraically closed field of positive characteristic such that  $(X, B)$  is strongly  $F$ -regular,  $A$  is ample and  $K_X + A + B$  is strictly nef, then  $K_X + A + B$  is ample. Similarly, we prove that for a log pair  $(X, A + B)$  with  $A$  being ample and  $B$  effective,  $K_X + A + B$  is big if it is nef and of maximal nef dimension. As an application, we establish a rationality theorem for the nef threshold and various results towards the minimal model program in dimension three in positive characteristic.

**RÉSUMÉ.** – Nous démontrons que, si  $(X, A + B)$  est une paire définie sur un corps algébriquement clos de caractéristique positive telle que  $(X, B)$  est fortement  $F$ -régulière,  $A$  est ample et  $K_X + A + B$  est strictement nef, alors  $K_X + A + B$  est ample. De la même manière, nous prouvons que, si  $(X, A + B)$  est une paire telle que  $A$  est ample et  $B$  est grand (« big »), alors une condition nécessaire et suffisante pour que le diviseur  $K_X + A + B$  soit grand est qu’il soit nef et de dimension nef maximale. Nous utilisons ces résultats pour démontrer un théorème de rationalité pour le seuil nef, ainsi que plusieurs résultats nécessaires au programme des modèles minimaux en caractéristique positive en dimension trois.

## 1. Introduction

One of the main objectives of the minimal model program is the study of the linear system associated to an adjoint divisor. For example, in characteristic 0, we have a good understanding of the linear system given by a multiple of a  $\mathbb{Q}$ -divisor  $L$  which is the sum of the canonical divisor and an ample  $\mathbb{Q}$ -divisor (e.g., see [18], [26], [5] and the references therein). A fundamental tool in birational geometry is Kawamata’s base point free theorem which asserts that if such a  $\mathbb{Q}$ -divisor  $L$  is nef then it is semiample (see [26]).

Because of the failure of the Kodaira vanishing theorem in positive characteristic, Kawamata’s base point free theorem and its generalizations are not known to hold in this case. The aim of this paper is to present a new approach to the base point free theorem in positive characteristic. We prove a special case of this result as well as several results which, in characteristic 0, are known to follow from the base point free theorem.

### 1.1. Strictly nef divisors

We first study strictly nef adjoint divisors, with possibly real coefficients. Recall that an  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor  $L$  on a proper variety  $X$  is said to be *strictly nef* if its intersection with any curve on  $X$  is positive. Mumford has constructed the first example of a strictly nef divisor which is not ample (see [15, Example 10]). See [29, Remark 3.2] for a similar example in positive characteristic. However, we show:

**THEOREM 1.1.** – *Let  $(X, B)$  be a strongly  $F$ -regular pair defined over an algebraically closed field  $k$  of characteristic  $p > 0$ , where  $B$  is an effective  $\mathbb{R}$ -divisor. Assume that  $A$  is an ample  $\mathbb{R}$ -divisor such that  $K_X + A + B$  is strictly nef. Then  $K_X + A + B$  is ample.*

From Theorem 1.1, we immediately obtain the following result:

**COROLLARY 1.2.** – *Let  $(X, \Delta)$  be a strongly  $F$ -regular projective pair with an effective  $\mathbb{R}$ -divisor  $\Delta$  over an algebraically closed field  $k$  of characteristic  $p > 0$  such that  $K_X + \Delta$  is big and strictly nef. Then  $K_X + \Delta$  is ample.*

In addition, we obtain the following result on the rationality of the nef threshold:

**THEOREM 1.3.** – *Let  $(X, B)$  be a strongly  $F$ -regular pair defined over an algebraically closed field of characteristic  $p > 0$ , where  $B$  is an effective  $\mathbb{Q}$ -divisor. Assume that  $K_X + B$  is not nef and  $A$  is an ample  $\mathbb{Q}$ -divisor. Let*

$$\lambda := \min\{t > 0 \mid K_X + B + tA \text{ is nef}\}.$$

*Then there exists a curve  $C$  in  $X$  such that  $(K_X + \lambda A + B) \cdot C = 0$ . In particular,  $\lambda$  is a rational number.*

When  $X$  is smooth and  $B = 0$ , the results follow from Mori's cone theorem [31]. We note that the assumption that  $(X, B)$  is strongly  $F$ -regular is analogous but more restrictive than the assumption that  $(X, B)$  is klt. In fact, in characteristic 0, all these statements are direct consequences of Kawamata's base point free theorem as we know that if  $(X, B)$  is a projective klt pair such that  $B$  is big and  $K_X + B$  is nef, then  $K_X + B$  is indeed semi-ample (see e.g., [26, 3.3]).

In positive characteristic, since [16] new techniques involving the Frobenius map have been developed to establish the positive characteristic analogs of many of the results, which in characteristic 0, are traditionally deduced from vanishing theorems. Very roughly, this is the general strategy that we follow in this paper as well.

On the other hand, the techniques used to prove the above results were inspired by an earlier attempt of the second author to prove Fujita's conjecture, which in turn was inspired by the proof of the effective base point free theorem in characteristic zero, by Angehrn and Siu [3]. In their paper, the authors construct zero-dimensional subschemes which are minimal log canonical centers for a suitable pair and using Nadel's vanishing theorems they are able to extend non-trivial sections to the whole variety. In positive characteristic, using the idea of twisting by Frobenius, the analogue would be to construct zero dimensional  $F$ -pure centers and use  $F$ -adjunction (see [34] for more details). Unfortunately there is a technical issue due to the index of the adjoint divisor, which we are not able to deal with, in general. Therefore, instead of using one divisor to cut the center, we study the trace map for all the powers of

Frobenius and assign different coefficients for each of these. For this reason, we introduce the use of *F-threshold functions* to replace the classical *F-pure threshold* and obtain a zero dimensional subscheme from which we can lift sections (see Subsection 3.1 and 3.2 for more details). Theorem 1.1 and Theorem 1.3 are proven in Section 4.

## 1.2. Divisors of maximal nef dimension

Using the same methods as above but cutting at two very general points, we study adjoint divisors of maximal nef dimension. More specifically, given a log pair  $(X, B)$  such that  $K_X + B$  is nef, the nef reduction map associated to  $K_X + B$  (see Subsection 2.4 for the definition) has proven to be a powerful tool to approach the Abundance conjecture (e.g., see [7, Section 9] and [2] for more details). Recall that a divisor over a proper variety  $X$  is said to be of *maximal nef dimension* if its intersection with any movable curve in  $X$  is positive (see Subsection 2.1 for the definition of movable curve). Thus, we obtain the following weak version of the base point free theorem:

**THEOREM 1.4.** – *Let  $X$  be a normal projective variety over an algebraically closed field of characteristic  $p > 0$ . Assume that  $A$  is an ample  $\mathbb{R}$ -divisor and  $B \geq 0$  is an  $\mathbb{R}$ -divisor such that  $K_X + B$  is  $\mathbb{R}$ -Cartier and  $K_X + A + B$  is nef and of maximal nef dimension. Then  $K_X + A + B$  is big.*

Note that the previous theorem does not require any assumption on the singularities of the pair  $(X, B)$ , nor on the coefficients of  $B$ .

As an application, we obtain the following result on the extremal ray associated to a nef but not big adjoint divisor:

**COROLLARY 1.5.** – *Let  $X$  be a normal projective variety, defined over an algebraically closed field of characteristic  $p > 0$ . Assume that  $A$  is an ample  $\mathbb{R}$ -divisor,  $B \geq 0$  is an  $\mathbb{R}$ -divisor such that  $K_X + B$  is  $\mathbb{R}$ -Cartier and  $L = K_X + A + B$  is nef and not big. Assume that*

$$\overline{NE}(X) \cap L^\perp = R$$

*is an extremal ray of  $\overline{NE}(X)$ .*

*Then  $X$  is covered by rational curves  $C$  such that  $[C] \in R$  and*

$$-(K_X + B) \cdot C \leq 2 \dim X.$$

Theorem 1.4 and Corollary 1.5 are proven in Section 5.

**REMARK 1.6.** – We were informed by J. McKernan that Theorem 1.4 and Corollary 1.5 were independently obtained by him using different methods [28].