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HYPERBOLIC COMPONENTS OF McMULLEN MAPS

BY Weiyuan QIU, Pascale ROESCH, Xiaoguang WANG
AND Yongcheng YIN

ABSTRACT. – In this article, we completely settle a question raised by B. Devaney. We prove that all the hyperbolic components are Jordan domains in the family of rational maps of McMullen type. Moreover, we give a precise description of all the rational maps on the outer boundary. It follows that the cusps are dense on the outer boundary.

RÉSUMÉ. – Dans cet article nous résolvons complètement une question posée par B. Devaney. Nous montrons que toutes les composantes hyperboliques sont des domaines de Jordan dans la famille de fractions rationnelles de type McMullen. De plus nous donnons une description précise de toutes les fractions du bord de la composante non bornée. Il en découle que les cusps sont denses dans le bord de la composante non bornée.

1. Introduction

In his article [22], Curt McMullen presented a family of rational maps with the particularity that, viewed as a dynamical system, it exhibits very rich dynamical behavior. Nevertheless, this family has a very simple form. It consists in a singular perturbation of the monomial $z \mapsto z^n$ acting on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$\begin{aligned} \widehat{\mathbb{C}} &\to \widehat{\mathbb{C}} \\ z &\mapsto z^n + \lambda z^{-m} \end{aligned}$$

where λ varies in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n, m \geq 1$.

The *Julia set*, which is the minimal totally invariant compact set of cardinality at least 3, appears in this family under several various classic fractals. Namely, Curt McMullen pointed out (in [22]) that when $(n, m) = (2, 3)$ and $\lambda \in \mathbb{C}^*$ is small, the Julia set is a Cantor set of circles. Moreover, in the hyperbolic case, the Julia set can also be homeomorphic to either

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a Cantor set, or a Sierpiński carpet as proven in [10]. A rational map is *hyperbolic* if every critical point converges under iterations to an attracting cycle.

The parameters $\lambda \in \mathbb{C}^*$ can then be divided into two classes, the hyperbolic ones and the others (see Figure 1, where hyperbolic parameters are in blue and yellow). Conjecturally the set of hyperbolic parameters (which is an open set) is dense in $\mathbb{C}^*$. In this article we study the boundaries of the hyperbolic components of the McMullen maps:

$$f_\lambda : z \mapsto z^n + \lambda z^{-n}, \;\; \lambda \in \mathbb{C}^*, \; n \geq 3.$$

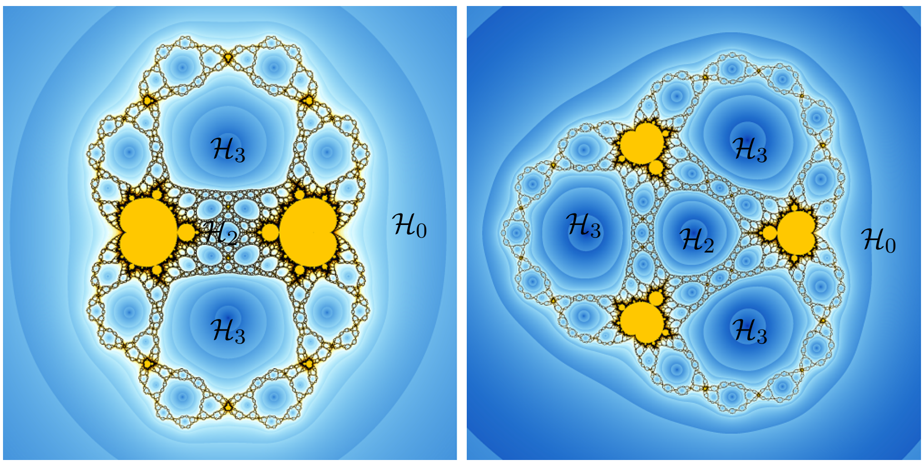

FIGURE 1. Parameter plane of McMullen maps, $n = 3, 4$.

Bob Devaney has proven in [6] that the boundary of the hyperbolic component containing the punctured neighborhood of the origin is a Jordan curve. He raised the question in 2004 at the Snowbird Conference (on the 25th Anniversary of the Mandelbrot set) whether all the other hyperbolic components of *escape type* (the free critical orbits escape to ∞) are Jordan domains.

The following Theorem 1.1 solves completely the question, it is the main result of the paper.

THEOREM 1.1. – *Fix any $n \geq 3$. The boundary of every hyperbolic component of the family $f_\lambda(z) = z^n + \lambda/z^n$, $\lambda \in \mathbb{C}^*$ is a Jordan curve.*

Moreover, we give a complete description of the dynamics of the McMullen maps lying at the boundary of the unbounded hyperbolic component $\mathcal{H}_0$. In this component, the Julia sets of the maps are Cantor sets, but at the boundary the Julia set is connected (see Section 2). As a corollary we obtain our second main result :

THEOREM 1.2. – *Cusps are dense in $\partial\mathcal{H}_0$.*

Here, according to McMullen [25], a parameter $\lambda \in \mathbb{C}^*$ is called a *cusp* if the map f_λ has a parabolic cycle. Geometrically, a cusp is a point where the bifurcation locus is cusp-shaped.

Let us recall that McMullen proved that cusps are dense on the Bers' boundary of Teichmüller space in [23]. The analogue of this result in the world of rational maps, as a conjecture posed by McMullen [25], would be, in the space of degree d polynomials, the density of geometrically finite parabolics on the boundary of the hyperbolic component $\mathcal{U}_d$ containing z^d. This conjecture is verified by P. Roesch [32] in the one-dimensional slice $z^d + cz^{d-1}$ ($d \geq 3$ and $c \in \mathbb{C}$). It remains open in full generality. The result we prove here is, in the sprit, related to this conjecture. But it has to be thought of as a phenomenon in some one-dimensional slices of rational maps.

We would like to explain why we concentrate on the case $m = n \geq 3$ of the general McMullen family $z \mapsto z^n + \lambda z^{-m},\ \lambda \in \mathbb{C}^*, m, n \geq 1$. This is because our proof rests on the technical Yoccoz puzzle theory. When $m = n \geq 3$, we can succeed in applying this theory to study both the dynamical plane and the parameter plane. However when $m = n = 2$, it is impossible to find a non-degenerate critical annulus for the Yoccoz puzzle constructed in [28]. The existence of a non-degenerate critical annulus is technically necessary in this theory. In the general case when $m \neq n \geq 1$, 0 and ∞ have different folds of symmetries. To the best of our knowledge, whether the Yoccoz puzzle structure exists in this case is unknown.

1.1. Overview of the paper

Let us recall some definitions and give the basic notions to precise our results before to go to the proof.

For any $\lambda \in \mathbb{C}^*$, the map f_λ has a fixed point at ∞ which is superattracting since the derivative is 0. The immediate attracting basin of ∞ denoted by B_λ is the set of points converging under iteration to ∞ and lying in the connected component of ∞. The set of critical points of f_λ is $\{0, \infty\} \cup C_\lambda$, where $C_\lambda = \{c \in \mathbb{C}; c^{2n} = \lambda\}$. Hence, there is only one free critical orbit (up to a sign) since besides ∞, there are only two critical values: $v_\lambda^+ = 2\sqrt{\lambda}$ and $v_\lambda^- = -2\sqrt{\lambda}$ (here, when restricted to the fundamental domain, v_λ^+ and v_λ^- are well-defined, see Section 3).

A rational map is *hyperbolic* if all critical orbits are attracted by the attracting cycles (see [27, 24]). Hence, a McMullen map f_λ is hyperbolic if the free critical orbit is attracted either by ∞ or by an attracting cycle in $\mathbb{C}$. Every hyperbolic component is isomorphic to either the unit disk $\mathbb{D}$ or $\mathbb{D}^* = \mathbb{D} - \{0\}$ (see Theorem 2.2). In particular, $\mathcal{H}_0$ is a topological punctured disk. Assuming Theorem 1.1, one gets a canonical parameterization $\nu : \mathbb{S} \to \partial\mathcal{H}_0$, where $\nu(\theta)$ is defined to be the landing point of the parameter ray $\mathcal{R}_0(\theta)$ (see Section 6) in $\mathcal{H}_0$. The complete characterization of $\partial\mathcal{H}_0$ we give is the following :

THEOREM 1.3 (Characterization of $\partial\mathcal{H}_0$ and cusps). – *We have*

1. $\lambda \in \partial\mathcal{H}_0$ *if and only if* ∂B_λ *contains either* C_λ *or a parabolic cycle.*
2. $\nu(\theta)$ *is a cusp if and only if* $n^p\theta \equiv \theta \bmod \mathbb{Z}$ *for some* $p \geq 1$.

Theorem 1.2 is an immediate consequence of Theorem 1.3 since the set $\{\theta \mid n^p\theta \equiv \theta \bmod \mathbb{Z}, p \geq 1\}$ is a dense subset of the unit circle $\mathbb{S}$.

The main part of the paper is to prove Theorem 1.1. We briefly sketch the idea of the proof and the organization of the paper.