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MORI CONES OF HOLOMORPHIC SYMPLECTIC VARIETIES OF K3 TYPE

BY Arend BAYER, Brendan HASSETT AND Yuri TSCHINKEL

Abstract. – We determine the Mori cone of holomorphic symplectic varieties deformation equivalent to the punctual Hilbert scheme on a K3 surface. Our description is given in terms of Markman's extended Hodge lattice.

Résumé. – On détermine les cônes de Mori des variétés symplectiques holomorphes qui se déforment au schéma de Hilbert de points sur une surface K3. Notre description est donnée en termes de structure de Hodge élargie de Markman.

Introduction

Let X be an irreducible holomorphic symplectic manifold. Let $(,)$ denote the Beauville-Bogomolov form on $H^2(X,\mathbb{Z})$; we may embed $H^2(X,\mathbb{Z})$ in $H_2(X,\mathbb{Z})$ via this form. Fix a polarization h on X; by a fundamental result of Huybrechts [17], X is projective if it admits a divisor class H with $(H,H) > 0$. It is expected that finer birational properties of X are also encoded by the Beauville-Bogomolov form and the Hodge structure on $H^2(X)$, along with appropriate extension data. In particular, natural cones appearing in the minimal model program—the moving cone, the nef cone, the pseudo-effective cone—should have a description in terms of this form.

Now assume X is deformation equivalent to the punctual Hilbert scheme $S^{[n]}$ of a K3 surface S with $n > 1$. Recall that

$$H^2(S^{[n]},\mathbb{Z})_{(,)} = H^2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}\delta, \quad (\delta,\delta) = -2(n-1) \tag{1}$$

where the restriction of the Beauville-Bogomolov form to the first factor is just the intersection form on S, and 2δ is the class of the locus of non-reduced subschemes. Recall from [20] that for K3 surfaces S, the cone of (pseudo-)effective divisors is the closed cone generated by

$$\{D \in \mathrm{Pic}(S) : (D,D) \geq -2, (D,h) > 0\}.$$

The first attempt to extend this to higher dimensions was [13]. Further work on moving cones was presented in [14, 24], which built on Markman's analysis of monodromy groups. The characterization of extremal rays arising from Lagrangian projective spaces $\mathbb{P}^n \hookrightarrow X$ has been addressed in [14, 12] and [3]. The paper [15] proposed a general framework describing all types of extremal rays; however, Markman found counterexamples in dimensions ≥ 10, presented in [5].

The formalism of Bridgeland stability conditions [7, 8] has led to breakthroughs in the birational geometry of moduli spaces of sheaves on surfaces. The case of punctual Hilbert schemes of $\mathbb{P}^2$ and del Pezzo surfaces was investigated by Arcara, Bertram, Coskun, and Huizenga [2, 16, 6, 10]. The effective cone on $(\mathbb{P}^2)^{[n]}$ has a beautiful and complex structure as n increases, which only becomes transparent in the language of stability conditions. Bayer and Macrì resolved the case of punctual Hilbert schemes and more general moduli spaces of sheaves on K3 surfaces [5, 4]. Abelian surfaces, whose moduli spaces of sheaves include generalized Kummer varieties, have been studied as well [31, 32].

In this note, we extend the results obtained for moduli spaces of sheaves over K3 surfaces to all holomorphic symplectic manifolds arising as deformations of punctual Hilbert schemes of K3 surfaces. Our principal result is Theorem 1 below, providing a description of the Mori cone (and thus dually of the nef cone).

In any given situation, this also leads to an effective method to determine the list of marked minimal models (i.e., birational maps $f : X \dashrightarrow Y$ where Y is also a holomorphic symplectic manifold): the movable cone has been described by Markman [23, Lemma 6.22]; by [14], it admits a wall-and-chamber decomposition whose walls are the orthogonal complements of extremal curves on birational models, and whose closed chambers correspond one-to-one to marked minimal model, as the pull-backs of the corresponding nef cones.

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1. Statement of results

Let X be deformation equivalent to the Hilbert scheme of length-n subschemes of a K3 surface. Markman, see [22, Theorem 1.10] and [23, Cor. 9.5], describes an extension of lattices

$$H^2(X, \mathbb{Z}) \subset \widetilde{\Lambda}$$

and weight-two Hodge filtrations

$$H^2(X, \mathbb{C}) \subset \widetilde{\Lambda}_{\mathbb{C}}$$

with the properties listed below. We will write

$$\theta_X : H^2(X) \subset \widetilde{\Lambda}_X$$

to denote the extension of Hodge structures with pairing; here θ_X is defined canonically up to a choice of sign.

- The orthogonal complement of $\theta_X\big(H^2(X,\mathbb{Z})\big)$ has rank one, and is generated by a primitive vector of square $2n-2$ and type $(1,1)$;
- as a lattice
$$\widetilde{\Lambda} \simeq U^4 \oplus (-E_8)^2$$
where U is the hyperbolic lattice and E_8 is the positive definite lattice associated with the corresponding Dynkin diagram;
- any parallel transport operator $\phi\colon H^2(X,\mathcal{Z}) \to H^2(X',\mathcal{Z})$ naturally lifts to an isometry of lattices $\tilde{\phi}\colon \widetilde{\Lambda}_X \to \widetilde{\Lambda}_{X'}$ such that
$$\tilde{\phi} \circ \theta_X = \theta_{X'} \circ \phi;$$
the induced action of the monodromy group on $\widetilde{\Lambda}/H^2(X,\mathbb{Z})$ is encoded by a character *cov* (see [21, Sec. 4.1]);
- we have the following Torelli-type statement: X_1 and X_2 are birational if and only if there is Hodge isometry
$$\widetilde{\Lambda}_{X_1} \simeq \widetilde{\Lambda}_{X_2}$$
taking $H^2(X_1,\mathbb{Z})$ isomorphically to $H^2(X_2,\mathbb{Z})$;
- if X is a moduli space $M_v(S)$ of sheaves (or of Bridgeland-stable complexes) over a K3 surface S with Mukai vector v then there is an isomorphism from $\widetilde{\Lambda}$ to the Mukai lattice of S taking $H^2(X,\mathbb{Z})$ to $v^\perp$.

Generally, we use v to denote a primitive generator for the orthogonal complement of $H^2(X,\mathbb{Z})$ in $\widetilde{\Lambda}$. Note that $v^2 = (v,v) = 2n-2$. When $X \simeq M_v(S)$ we may take the Mukai vector v as the generator.

As the dual of θ_X we obtain a homomorphism(1)

$$\theta_X^\vee : \widetilde{\Lambda}_X \twoheadrightarrow H_2(X,\mathbb{Z})$$

which restricts to an inclusion

$$H^2(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$$

of finite index. By extension, it induces a $\mathbb{Q}$-valued Beauville-Bogomolov form on $H_2(X,\mathbb{Z})$.

Assume X is projective. Let $H^2(X)_{\mathrm{alg}} \subset H^2(X,\mathbb{Z})$ and $\widetilde{\Lambda}_{\mathrm{alg}} \subset \widetilde{\Lambda}_X$ denote the algebraic classes, i.e., the integral classes of type $(1,1)$. Since the orthogonal complement of $i_X\big(H^2(X)\big)$ is generated by an algebraic class, it follows dually that $a \in \widetilde{\Lambda}_X$ is of type $(1,1)$ if and only if $\theta^\vee(a)$ is. The Beauville-Bogomolov form on $H^2(X)_{\mathrm{alg}}$ has signature $(1,\rho(X)-1)$, where $\rho(X) = \dim(H^2_{\mathrm{alg}}(X))$. The *Mori cone* of X is defined as the closed cone in $H_2(X,\mathbb{R})_{\mathrm{alg}}$ containing the classes of algebraic curves in X. The *positive cone* (or more accurately, non-negative cone) in $H^2(X,\mathbb{R})_{\mathrm{alg}}$ is the closure of the connected component of the cone

$$\{D \in H^2(X,\mathbb{R})_{\mathrm{alg}} : D^2 > 0\}$$

(1) We will often drop the subscript X from the notation when the context is clear.