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# DIMENSIONS OF SPACES OF LEVEL ONE AUTOMORPHIC FORMS FOR SPLIT CLASSICAL GROUPS USING THE TRACE FORMULA

BY Olivier TAÏBI

Abstract. – We derive explicit formulae for the number of level one, regular algebraic and essentially self-dual automorphic cuspidal representations of general linear groups over $\mathbb{Q}$, as a function of the Hodge weights. As a consequence, we obtain formulae for dimensions of spaces of vector-valued Siegel modular cusp forms.

Résumé. – Nous démontrons des formules explicites pour le nombre de représentations automorphes cuspidales algébriques régulières et essentiellement auto-duales pour les groupes linéaires sur $\mathbb{Q}$, comme fonction des poids de Hodge. Nous en déduisons des formules explicites pour les dimensions des espaces de formes modulaires de Siegel cuspidales à valeurs vectorielles.

## 1. Introduction

Using Arthur's trace formula in [5] and Arthur's endoscopic classification of the discrete spectrum for special orthogonal and symplectic groups in [8], we give an algorithm to derive explicit formulae counting the number of level one, regular algebraic and essentially self-dual automorphic cuspidal representations of general linear groups, as a function of the Hodge weights. Before elaborating more on our method in this introduction, we state two problems that motivate this work:

- giving explicit dimension formulae for vector-valued Siegel modular forms, a problem which was open for genera greater than 2,
- classifying motives of conductor 1 (or good reduction) and given Hodge weights.

The major part of this work was done while the author was a doctoral student at École polytechnique, Palaiseau, and employed by École Normale Supérieure, Paris. This work was completed while the author was a research associate at Imperial College London, supported by ERC Starting Grant 306326.

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### 1.1. Two problems

1.1.1. *Dimensions of spaces of Siegel cusp forms.* – The first problem is a classical one: explicitly determining the dimensions of spaces of vector-valued Siegel modular forms in genus $n \geq 1$ (often called "degree" in the literature). We will only consider this problem in level one, i.e., for the full modular group $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$. Given integers $k_1 \geq \cdots \geq k_n$, let $r$ be the holomorphic (equivalently, algebraic) finite-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight $\underline{k} = (k_1, \ldots, k_n)$ and let $S_r(\Gamma_n) = S_{\underline{k}}(\Gamma_n) = S_{k_1,\ldots,k_n}(\Gamma_n)$ denote the space of Siegel cusp forms of genus $n$, level $\Gamma_n$ and weight $r$.

For $n = 1$, it is well-known that the graded $\mathbb{C}$-algebra of modular forms is freely generated by the Eisenstein series $E_4$ and $E_6$. This implies that for $k > 1$,

$$\dim S_{2k}(\Gamma_1) = \begin{cases} \lfloor k/6 \rfloor & \text{if } k \neq 1 \bmod 6 \\ \lfloor k/6 \rfloor - 1 & \text{if } k = 1 \bmod 6 \end{cases}$$
$$= \frac{k}{6} - \frac{7}{12} + \frac{(-1)^k}{4} + \mathrm{tr}_{\mathbb{Q}(j)/\mathbb{Q}}\left(\frac{(2+j)j^k}{9}\right)$$

where $j^2 + j + 1 = 0$. Together with the fact that $S_0(\Gamma_1) = S_2(\Gamma_1) = 0$, this is equivalent to

$$\sum_{k \geq 0} t^k \dim S_k(\Gamma_1) = \frac{1}{(1-t^4)(1-t^6)} - \frac{1}{1-t^2} + t^2. \qquad (1.1.1)$$

In genus 2, Igusa [49] determined the structure of the ring of *scalar* (i.e., $k_1 = k_2$) Siegel modular forms and its ideal of cusp forms, which implies a dimension formula of a similar kind, equivalent to:

$$\sum_{k \geq 0} t^k \dim S_{k,k}(\Gamma_2) = \frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} - \frac{1}{(1-t^4)(1-t^6)}.$$

Tsushima [89, Theorem 4] later gave a formula for the dimension of $S_{k_1,k_2}(\Gamma_2)$ for $k_1 > k_2 \geq 5$ using the holomorphic Lefschetz formula of Atiyah-Singer and the Kawamata-Viehweg vanishing theorem. Recently Petersen [76] has shown that Tsushima's formula also holds for $k_1 > k_2 \geq 3$, as conjectured by Tsushima (for $k_2 = 4$) and Ibukiyama (for $k_2 = 3$, this is particular to the case of full level $\Gamma_2$). The method used in the present paper also implies this result. In genus 3 Tsuyumine [91, p. 831] determined the structure of the ring of scalar Siegel modular forms and its ideal of cusp forms, and thus obtained an explicit formula for $\sum_{k \geq 0} t^k \dim S_{k,k,k}(\Gamma_3)$. More recently Bergström, Faber and van der Geer studied the cohomology of certain local systems on the moduli space $\mathscr{A}_3$ of principally polarized abelian threefolds, and conjectured a formula for the Euler-Poincaré characteristic of its cohomology (as a motive) in terms of Siegel modular forms. They were able to derive a conjectural formula for $\dim S_{\underline{k}}(\Gamma_3)$ for $k_3 \geq 4$ and $\underline{k} \neq (4,4,4)$ ([10, Conjecture 7.3]).

One of the goals of this paper is to prove this conjecture and to generalize these explicit formulae to higher genera; in particular we will prove the following

THEOREM A (Dimension formula for spaces of Siegel cusp forms)

Let $n \geq 1$. For $m \geq 1$ denote $\zeta_m = \exp(2i\pi/m)$. There exists a finite family $(m_a, P_a, \Lambda_a)_{a \in A}$, which we make explicit for all $n \leq 7$, where for any $a \in A$

– $m_a \geq 1$ is an integer,
– $P_a \in \mathbb{Q}(\zeta_{m_a})[X_1, \dots, X_n]$,
– $\Lambda_a : (\mathbb{Z}/m_a\mathbb{Z})^n \to \mathbb{Z}/m_a\mathbb{Z}$ is a surjective group morphism,

such that for any $k_1 \geq k_2 \geq \cdots \geq k_n > n+1$, we have

$$\dim S_{\underline{k}}(\Gamma_n) = \sum_{a \in A} \mathrm{tr}_{\mathbb{Q}(\zeta_{m_a})/\mathbb{Q}} \left( P_a(k_1, \dots, k_n) \zeta_{m_a}^{\Lambda_a(k_1, \dots, k_n)} \right). \tag{1.1.2}$$

For $n = 3$, the first genus for which we obtain a new result, this explicit formula has 370 terms (i.e., $|A| = 370$), and thus it is too long to print.

We have expressed the formula in the most elementary manner, but in fact the family $(m_a, P_a, \Lambda_a)_{a \in A}$ is not exactly what we compute. More precisely, the right hand side of 1.1.2 is equal to a linear combination with rational coefficients of traces, in algebraic representations determined by $\underline{k}$, of rational torsion elements of split classical groups of rank $\leq n$. What we give is an algorithm to compute these rational coefficients, which certainly deserve to be called "masses". Formula 1.1.2 can then be derived using (an extension to singular elements of) Weyl's character formula. The algorithm works for any $n$, but our computer was only able to calculate these masses for $n \leq 7$.

As we will recall in Section 5, the weights $k_1 \geq \cdots \geq k_n$ corresponding to holomorphic discrete series for $\mathbf{PGSp}_{2n}(\mathbb{R})$ are those such that $k_n \geq n+1$. It is also possible to compute $\dim S_{\underline{k}}(\Gamma_n)$ when $k_n = n+1$, but the resulting formula is not the specialization of the right hand side of 1.1.2, as the case $n = 1$ (Formula 1.1.1) already shows. Our method does not allow us to compute the dimensions for weights such that $k_n \leq n$. The values for $\dim S_{\underline{k}}(\Gamma_n)$ for $n \leq 7$ and $16 \geq k_1 \geq \cdots \geq k_n \geq n+1$ are available at `http://wwwf.imperial.ac.uk/~otaibi/dimtrace`. See the table in Section 5.5 for values in the scalar case $k_1 = \cdots = k_n$.

Our endoscopic method is not as direct as Tsushima's or as using the trace formula directly with a pseudo-coefficient of holomorphic discrete series at the real place, but we will see that it gives much more information than just the dimension. In particular, it distinguishes between eigenforms which are endoscopic liftings from lower rank groups (e.g., Duke-Imamoğlu-Ikeda liftings, see [50]) and "genuinely new" eigenforms. As a corollary of our exposition and [18] or [29], we have that for $k_1 > \cdots > k_n > n+1$, all the eigenforms in $S_{\underline{k}}(\Gamma_n)$ satisfy the Ramanujan conjecture.

We hope that these dimension formulae will be used to prove structure theorems for rings of scalar modular forms, ideals of cusp forms and modules of vector-valued forms, and to study the geometry of the moduli stack $\mathcal{A}_g$.

1.1.2. *Motives over $\mathbb{Q}$ with good reduction.* – The second problem that motivates this work stems from Minkowski's theorem stating that there is no non-trivial finite extension of $\mathbb{Q}$ unramified at all primes. From the point of view of arithmetic geometry, a natural generalization would be to classify smooth proper schemes $X$ over $\mathbb{Z}$ with certain properties. For Example Minkowski's theorem can be restated as follows: any proper smooth $X \to \mathrm{Spec}(\mathbb{Z})$ of relative dimension 0 is a disjoint union of finitely many copies of $\mathrm{Spec}(\mathbb{Z})$. A celebrated result in this direction is the proof by Fontaine [36] and independently Abrashkin [1] of Shavarevich's conjecture that there are no non-trivial abelian varieties over $\mathbb{Z}$. Even for a fixed relative dimension, it is certainly too ambitious to ask for a classification of all proper