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A SCATTERING THEORY FOR THE WAVE EQUATION ON KERR BLACK HOLE EXTERIORS

BY Mihalis DAFERMOS, Igor RODNIANSKI
AND Yakov SHLAPENTOKH-ROTHMAN

Abstract. – We develop a definitive physical-space scattering theory for the scalar wave equation $\Box_g \psi = 0$ on Kerr exterior backgrounds in the general subextremal case $|a| < M$. In particular, we prove results corresponding to "existence and uniqueness of scattering states" and "asymptotic completeness" and we show moreover that the resulting "scattering matrix" mapping radiation fields on the past horizon $\mathcal{H}^-$ and past null infinity $\mathcal{I}^-$ to radiation fields on $\mathcal{H}^+$ and $\mathcal{I}^+$ is a bounded operator. The latter allows us to give a time-domain theory of superradiant reflection. The boundedness of the scattering matrix shows in particular that the maximal amplification of solutions associated to ingoing finite-energy wave packets on past null infinity $\mathcal{I}^-$ is bounded. On the frequency side, this corresponds to the novel statement that the suitably normalized reflection and transmission coefficients are uniformly bounded independently of the frequency parameters. We further complement this with a demonstration that superradiant reflection indeed amplifies the energy radiated to future null infinity $\mathcal{I}^+$ of suitable wave-packets as above. The results make essential use of a refinement of our recent proof [30] of boundedness and decay for solutions of the Cauchy problem so as to apply in the class of solutions where only a degenerate energy is assumed finite. We show in contrast that the analogous scattering maps cannot be defined for the class of finite non-degenerate energy solutions. This is due to the fact that the celebrated horizon red-shift effect acts as a blue-shift instability when solving the wave equation backwards.

Résumé. – Nous développons une théorie de la diffusion définitive en espace physique pour l'équation scalaire d'onde dans la région extérieure de la métrique de Kerr dans le cas sous-extrémal général $|a| < M$. En particulier, nous prouvons des résultats qui correspondent à « l'existence et l'unicité des états de diffusion » et la « complétude asymptotique » et nous montrons de plus que la matrice de diffusion qui envoie les champs de radiation sur l'horizon passé et l'infini nul passé aux champs sur l'horizon futur et l'infini nul futur est un opérateur borné. Ce dernier point nous permet de donner une théorie de réflexion superradiante dans le domaine temporel. Le fait que la matrice de diffusion est bornée montre en particulier que l'amplification maximale de solutions associées aux paquets d'ondes entrants d'énergie finie sur l'infini nul passé est bornée. En fréquence, cela correspond à l'affirmation nouvelle que les coefficients de réflexion et de transmission, convenablement normalisés, sont bornés uniformément, indépendamment des paramètres de fréquence. Nous complétons ceci de plus avec une démonstration que la réflexion superradiante amplifie effectivement l'énergie rayonnée à l'infini nul futur, pour les paquets d'ondes appropriés comme ci-dessus. Les résultats font usage essentiel d'un raffinement de notre démonstration récente [30] de la bornitude et de la décroissance

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des solutions du problème de Cauchy de façon à s'appliquer à la classe de solutions où seulement une énergie dégénérée est supposée finie. Nous montrons en contraste que l'application de diffusion analogue ne peut pas être définie pour la classe de solutions d'énergie finie non dégénérée. C'est dû au fait que le célèbre effet de décalage vers le rouge agit comme une instabilité de décalage vers le bleu quand on résout l'équation d'onde rétrograde.

1. Introduction

Black holes play a central role in our present general relativistic picture of the universe. At the same time, however, they are perhaps the example *par excellence* of a physical object which cannot be observed "directly". An effective approach to infer both the very presence but also the finer properties of black holes proceeds through the study of the scattering of waves on their exterior. Hence, a theoretical understanding of scattering theory in this context is of paramount importance.

The bulk of the now classical black hole scattering-theory literature concerns only the *fixed-frequency* study of solutions $u_{(\omega,m,\ell)}(r^*)$ to the radial o.d.e.

$$u'' + \omega^2 u = Vu, \tag{1}$$

where $V = V_{(\omega,m,\ell)}(r^*)$, resulting from Carter's remarkable separation [15] of the linear scalar wave equation

$$\Box_g \psi = 0 \tag{2}$$

on Kerr black hole backgrounds $(\mathcal{M}, g_{a,M})$. One can also consider more complicated systems like the Maxwell Equations or the equations of linearised gravity. See Chandrasekhar's monumental [16] and the monograph [40].

Beyond formal fixed-frequency statements concerning (1), true scattering results in the "time-domain," describing actual *finite-energy* solutions of (2) and related equations, have only been obtained in various special cases. Let us already mention the pioneering results of Dimock and Kay [33, 35, 34] in the Schwarzschild $a = 0$ case. See also [8, 9]. In the case of rotating Kerr black holes with $a \neq 0$, on the other hand, despite recent progress on the Cauchy problem, first for the $|a| \ll M$ case [28, 4, 69] and then, for the full subextremal range $|a| < M$ in [30], the most basic questions of scattering theory for (2) have remained to this day unanswered. In particular:

(a) Can one associate a finite-energy solution of (2) to every suitable finite-energy past/future asymptotic state? *(Existence of scattering states.)*
(b) Is the above association unique, i.e., do two finite-energy solutions having the same asymptotic state necessarily coincide? *(Uniqueness of scattering states.)*
(c) Do the above solutions parametrised by finite-energy past/future asymptotic states describe the totality of finite-energy solutions ψ to (2)? *(Asymptotic completeness.)*

See the classic [62] for a general introduction to the scattering theory framework in physics.

At the conceptual level, one of the most interesting new phenomena of black hole scattering which arises when passing from the Schwarzschild $a = 0$ to the rotating $a \neq 0$ Kerr case is that of *superradiance*. This already can be seen at the level of the fixed-frequency o.d.e. (1). We review this very quickly for the benefit of the reader familiar with the classical physics literature [16]. (1) For each fixed frequency triple (ω, m, ℓ) with $\omega \in \mathbb{R}$, one can define two complex-valued solutions $U_{\rm hor}(r^*)$ and $U_{\rm inf}(r^*)$ of (1) so that

$$U_{\rm hor} \sim e^{-i(\omega-\omega_+ m)r^*} \text{ as } r^* \to -\infty, \qquad U_{\rm inf} \sim e^{i\omega r^*} \text{ as } r^* \to \infty,$$

corresponding to the asymptotic behavior of the potential V, which is itself real. Here ω_+ is related to the Kerr parameters a, M by the formula $2M\omega_+(M + \sqrt{M^2 - a^2}) = a$. The linear independence of $U_{\rm hor}$ and $U_{\rm inf}$ is the statement of *mode stability* on the real axis and was proven recently by one of us [67], extending the transformation theory of [73]. By dimensional considerations, this linear independence at one go answers the "fixed frequency" analog of questions (a)–(c) in the affirmative. It follows that since $\overline{U_{\rm inf}}$ also solves (1), we may write

$$-\frac{\omega\mathfrak{T}}{(\omega - \omega_+ m)} U_{\rm hor} = \mathfrak{R} U_{\rm inf} + \overline{U_{\rm inf}}, \tag{3}$$

where $\mathfrak{T} = \mathfrak{T}(\omega, m, \ell)$ and $\mathfrak{R} = \mathfrak{R}(\omega, m, \ell)$ are known as the *transmission* and *reflexion* coefficients. Formally, these coefficients describe the proportion of "energy" at fixed frequency (ω, m, ℓ) transmitted to the horizon and reflected to infinity, respectively, of purely incoming wave from past infinity. With the precise normalization of (3), which will be in fact motivated by the considerations of this paper, the energy identity associated to (1) yields

$$|\mathfrak{R}|^2 + \frac{\omega}{\omega - \omega_+ m}|\mathfrak{T}|^2 = 1. \tag{4}$$

Superradiance, first discussed by Zeldovich [74], corresponds to the fact that, for the frequency range

$$\omega(\omega - \omega_+ m)^{-1} < 0, \tag{5}$$

the transmission coefficient $\mathfrak{T}$ is weighted with a negative sign in (4) allowing thus the reflection coefficient $\mathfrak{R}$ to have norm strictly greater than 1

$$|\mathfrak{R}(\omega, m, \ell)| > 1. \tag{6}$$

That is to say, there is a nontrivial energy amplification factor at fixed frequency. The first estimates for the maximum reflection coefficient in various frequency regimes go back to pioneering work of Starobinskii [68] (see also [70]), but even the statement of the uniform boundedness of $\mathfrak{R}(\omega, m, \ell)$ over all superradiant frequencies (5) has remained an open problem.

In passing from a fixed-frequency scattering theory to a true time-domain scattering theory, the absence of an obvious quantitative frequency-independent control of the coefficient $\mathfrak{R}(\omega, m, \ell)$ presents itself as a fundamental difficulty. Moreover, an additional difficulty is identifying the correct notion of "energy" with respect to which solutions should be defined. In particular, one requires a notion of energy which controls solutions of (2) not

(1) All notations here will be explained in detail in the paper. The reader for which this is unfamiliar can skip directly to the next paragraph!