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ALBANESE MAPS OF PROJECTIVE MANIFOLDS WITH NEF ANTICANONICAL BUNDLES

BY JUNYAN CAO

ABSTRACT. – Let X be a projective manifold such that the anticanonical bundle $-K_X$ is nef. We prove that the Albanese map $p : X \to Y$ is locally trivial. In particular, p is a submersion.

RÉSUMÉ. – Soit X une variété projective à fibré anticanonique nef. On montre que l'application d'Albanese $p : X \to Y$ est localement triviale. En particulier, p est lisse.

1. Introduction

Let X be a compact Kähler manifold such that the anticanonical bundle $-K_X$ is nef, and let $p : X \to Y$ be the Albanese map. By the work of Q. Zhang [40] and M. Păun [32], we know that π is a fibration, i.e., π is surjective and has connected fibers. Conjecturally, the Albanese map has more regularities:

CONJECTURE 1.1 ([17]). – *Let X be a compact Kähler manifold such that $-K_X$ is nef, and let $p : X \to Y$ be the Albanese map. Then p is locally trivial, i.e., for any small open set $U \subset Y$, $p^{-1}(U)$ is biholomorphic to the product $U \times F$, where F is the generic fiber of p. In particular, p is a submersion.*

This conjecture has been proved under the stronger assumption that T_X is nef, $-K_X$ is hermitian or the anticanonical bundle of the generic fiber is big [7, 16, 17, 6, 10]. For the general case, [27] proved that p is equidimensional and has reduced fibers. In low dimension, [34] proved that the Albanese map is a submersion for 3-dimensional projective manifolds.

The aim of this article is to prove the conjecture under the assumption that X is projective:

THEOREM 1.2. – *Let X be a projective manifold with nef anti-canonical bundle and let $p : X \to Y$ be the Albanese map. Then p is locally trivial, i.e., for any small open set $U \subset Y$, $p^{-1}(U)$ is biholomorphic to the product $U \times F$, where F is the generic fiber of p.*

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As an application of Theorem 1.2, we can study the structure of the universal cover of projective manifolds with nef anticanonical bundles. Recalling that, for a compact Kähler manifold with hermitian semipositive anticanonical bundle X, [17, 6] proved that the universal covering $\widetilde{X}$ admits a holomorphic and isometric splitting

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_l,$$

where Y_j are irreducible Calabi-Yau manifolds, S_k are irreducible hyperkähler manifolds and Z_l are rationally connected manifolds with irreducible holonomy. They expect a similar splitting result for compact Kähler manifolds with nef anticanonical bundles [(1)]:

CONJECTURE 1.3. – *Let X be a compact Kähler manifold with nef anticanonical bundle. Then the universal covering $\widetilde{X}$ of X admits the following splitting*

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times Z,$$

where Y_j are irreducible Calabi-Yau manifolds, S_k are irreducible hyperkähler manifolds and Z is a rationally connected manifold.

This conjecture was proved for 3-dimensional projective manifolds [1]. For an arbitrary compact Kähler manifold X with nef anticanonical bundle, thanks to [5, 31, 32], we know that the fundamental group $\pi_1(X)$ of X is almost abelian (cf. also Proposition 4.3). Together with Theorem 1.2, we get the following partial result for Conjecture 1.3.

COROLLARY 1.4. – *Let X be a projective manifold with nef anticanonical bundle. Then the universal cover $\widetilde{X}$ of X admits the following splitting*

$$\widetilde{X} \simeq \mathbb{C}^r \times F.$$

Here F is a compact simply connected projective manifold with nef anticanonical bundle, and $r = \sup h^{1,0}(\widehat{X})$ where the supremum is taken over all finite étale covers $\widehat{X} \to X$.

Let us explain briefly the basic ideas of the proof of Theorem 1.2. Like many works on the study of the manifolds with nef anticanonical bundles (cf. [4, 10, 8, 13, 15, 19, 21, 27, 30, 32, 41] to quote only a few), the proof of Theorem 1.2 is based on the positivity of direct images. More precisely, in the setting of Theorem 1.2, let L be a pseudo-effective line bundle on X and let A be an ample line bundle on X. In general, we don't know about the positivity of $p_\star(L+A)$. However, as $-K_{X/Y}$ is nef in our case, we can obtain the positivity of $p_\star(L+A)$ by using the following very elegant argument in [41].

Fix a possibly singular metric h_L such that $i\Theta_{h_L}(L) \geq 0$ in the sense of current and let $m \in \mathbb{N}$ large enough such that $\mathscr{I}(h_L^{\frac{1}{m}}) = \mathcal{O}_X$ [(2)]. We have

$$L + A = mK_{X/Y} + (-mK_{X/Y} + A) + L. \tag{1}$$

As $-K_{X/Y}$ is nef, $(-mK_{X/Y} + A)$ is ample and can be equipped with a smooth metric h_1 with positive curvature. Therefore $h = h_1 + h_L$ defines a possibly singular metric on

$$\widetilde{L} := (-mK_{X/Y} + A) + L \tag{2}$$

(1) Very recently, [11] proved the conjecture for projective manifolds with nef anticanonical bundles.

(2) We refer to the paragraph before Theorem 2.6 for the definition of $\mathscr{I}(h_L^{\frac{1}{m}})$.

with $i\Theta_h(\widetilde{L}) \geq 0$ and $\mathcal{J}(h^{\frac{1}{m}}) = \mathcal{O}_X$. Then the powerful results on the positivity of direct images (cf.[2, 3, 23, 24, 25, 20, 33, 36, 39] among many others) can be used to study the direct image

$$p_\star(mK_{X/Y} + \widetilde{L}) = p_\star(L + A).$$

We refer to Proposition 2.9 and Corollary 2.10 for some more accurate statements.

Another main ingredient involved in the proof is inspired and very close to [16, 3.D] and [8]. Recalling that, under the assumption that $-K_{X/Y}$ is p-ample, [16] proved that $p_\star(-mK_{X/Y})$ is numerically flat for every $m \in \mathbb{N}$. Thanks to this numerically flatness, we can prove the local trivialness of the Albanese map [16, 10]. In the situation of Theorem 1.2, as $-K_{X/Y}$ is not necessarily strictly positive along the fibers, we consider an arbitrary p-ample line bundle L on X to replace $-K_{X/Y}$. By [27], we can assume that $p_\star(mL)$ is locally free for every $m \in \mathbb{N}$. By combining [16, 3.D] with the positivity of direct images discussed above, we can prove that, $p_\star(mL')$ is numerically flat for every $m \in \mathbb{N}$, where $L' := \operatorname{rank} p_\star(L) \cdot L - p^\star \det p_\star(L)$. The fibration p is thus locally trivial by using a criteria proved in [16, 10], cf. also Proposition 2.4.

Here are the main steps of the proof of Theorem 1.2. Firstly, using the positivity of direct images [3], the diagonal method of Viehweg [39, Thm 6.24] as well as the method of Zhang [41], we prove in Proposition 3.1 that for any p-ample line bundle A on X, if $p_\star(A)$ is locally free, then $rA - p^\star \det p_\star(A)$ is pseudo-effective, where r is the rank of $p_\star(A)$. Secondly, after passing to some isogeny of the abelian variety Y, we can assume that $\frac{1}{r} \det p_\star(A)$ is a line bundle. By using an isogeny argument [16, Lemma 3.21] and [3], we prove that $p_\star(A) \otimes (-\frac{1}{r} \det p_\star(A))$ is numerically flat. Finally, we use the arguments in [16, 10] to conclude that p is locally trivial.

Our paper is organized as follows. In Section 2, after recalling some basic notations and results about the positivity of line bundles and vector bundles, we will review a criteria of the locally trivialness in [10]. We will also gather some results about the positivity of direct images in [2, 3, 33]. In Section 3, inspired by [16, Section 3.D], we will prove two important propositions which will be the key ingredients in the proof of main Theorem 1.2. Both propositions imply in particular that the Albanese map is very rigid. Finally, a complete proof of Theorem 1.2 and Corollary 1.4 is provided in Section 4.

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2. Preparation

We first recall some basic notations about the positivity of line bundles and vector bundles. We refer to [14, 16, 26] for more details.

Definition 2.1. – *Let X be a projective manifold.*