ISSN 0012-9593

quatrième série - tome 52 fascicule 1 janvier-février 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Florent BERTHELIN & Julien VOVELLE

Stochastic isentropic Euler equations

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 52, 2019, p. 181 à 254

STOCHASTIC ISENTROPIC EULER EQUATIONS

BY FLORENT BERTHELIN AND JULIEN VOVELLE

ABSTRACT. – We study the stochastically forced system of isentropic Euler equations of gas dynamics with a γ-law for the pressure. We show the existence of martingale weak entropy solutions; we also discuss the existence and characterization of invariant measures in the concluding section.

RÉSUMÉ. – Nous étudions le système d'Euler des gaz isentropiques, pour une loi de pression en ρ^γ, avec un forçage stochastique. Nous prouvons l'existence de solutions martingales vérifiant des inégalités entropiques. Nous discutons également de l'existence et de la caractérisation de mesures invariantes dans la section de conclusion.

1. Introduction

In this paper, we study the stochastically forced system of isentropic Euler equations of gas dynamics with a γ-law for the pressure.

Let $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t), (\beta_k(t)))$ be a stochastic basis, let $\mathbb{T}$ be the one-dimensional torus, let $T > 0$ and set $Q_T := \mathbb{T} \times (0, T)$. We study the system

$$d\rho + \partial_x(\rho u)dt = 0, \qquad \text{in } Q_T, \tag{1.1a}$$

$$d(\rho u) + \partial_x(\rho u^2 + p(\rho))dt = \Phi(\rho, u)dW(t), \qquad \text{in } Q_T, \tag{1.1b}$$

$$\rho = \rho_0, \quad \rho u = \rho_0 u_0, \qquad \text{in } \mathbb{T} \times \{0\}, \tag{1.1c}$$

where p follows the γ-law

$$p(\rho) = \kappa \rho^\gamma, \quad \kappa = \frac{\theta^2}{\gamma}, \quad \theta = \frac{\gamma - 1}{2}, \tag{1.2}$$

for $\gamma > 1$, W is a cylindrical Wiener process and $\Phi(0, u) = 0$. Therefore the noise affects the momentum equation only and vanishes in vacuum regions. Our aim is to prove the existence of solutions to (1.1) for general initial data (including vacuum), cf. Theorem 2.1 below.

The second author is partially supported by ANR STOSYMAP, ANR STAB and ERC NuSiKiMo.

 doi:10.24033/asens.2386

There are to our knowledge no existing results on stochastically forced systems of first-order conservation laws, with the exception of the papers by Kim, [50], and Audusse, Boyaval, Goutal, Jodeau, Ung, [2]. In [50] the problematic is the possibility of global existence of *regular* solutions to symmetric hyperbolic systems under suitable assumptions on the structure of the stochastic forcing term. In [2] is derived a shallow water system with a stochastic Exner equation as a model for the dynamics of sedimentary river beds. On second-order stochastic systems, and specifically on the stochastic compressible Navier-Stokes equation (1), different results have been obtained recently, see the papers by Breit, Feireisl, Hofmanová, Maslowski, Novotny, Smith, [36, 11, 10, 66] (see also the older work by Tornare and Fujita, [68]).

The *incompressible* Euler equations with stochastic forcing terms have been studied in particular by Bessaih, Flandoli, [5, 8, 6, 7], Capiński, Cutland, [16], Brzeźniak, Peszat, [14], Cruzeiro, Flandoli, Malliavin, [24], Brzeźniak, Flandoli, Maurelli, [12], Glatt-Holtz and Vicol, [41], Cruzeiro and Torrecilla, [25]. We refer in particular to [41] for results in space dimension 3.

In the deterministic case, and in space dimension 1, the existence of weak entropy solutions to the isentropic Euler system has been proved by Lions, Perthame, Souganidis in [54]. Let us mention also the anterior papers by Di Perna [32], Ding, Chen, Luo [31], Chen [20], Lions, Perthame, Tadmor [58]. The uniqueness of weak entropy solutions is still an open question.

For *scalar* non-linear hyperbolic equations with a stochastic forcing term, the theory has recently known a lot of developments. Well-posedness has been proved in different contexts and under different hypotheses and also with different techniques: by Lax-Oleinik formula (E, Khanin, Mazel, Sinai [35]), Kruzhkov doubling of variables for entropy solutions (Kim [51], Feng, Nualart [37], Vallet, Wittbold [70], Chen, Ding, Karlsen [21], Bauzet, Vallet, Wittbold [4]), kinetic formulation (Debussche, Vovelle [28, 29]). Resolution in L^1 has been given in [30]. Let us also mention the works of Hofmanová in this fields (extension to second-order scalar degenerate equations, convergence of the BGK approximation [45, 27, 46]) and the recent works by Hofmanová, Gess, Lions, Perthame, Souganidis [56, 55, 57, 39, 40, 47] on scalar conservation laws with quasilinear stochastic terms.

We will show existence of martingale solutions to (1.1), see Theorem 2.1 below. The procedure is standard: we prove the convergence of (subsequence of) solutions to the parabolic approximation to (1.1). For this purpose we have to adapt the concentration compactness technique (cf. [32, 54]) of the deterministic case to the stochastic case. Such an extension has already been done for scalar conservation laws by Feng and Nualart [37] and what we do is quite similar. The mode of convergence for which there is compactness, if we restrict ourselves to the sample variable ω, is the convergence in law. That is why we obtain martingale solutions. There is a usual trick, the Gyöngy-Krylov characterization of convergence in probability, that allows to recover pathwise solutions once pathwise uniqueness of solutions is known (cf. [43]). However for the stochastic problem (1.1) (as it is already the case for the deterministic one), no such results of uniqueness are known.

(1) Which, to be exact, is first-order in the density and second-order in the velocity.

A large part of our analysis is devoted to the proof of existence of solutions to the parabolic approximation. What is challenging and more difficult than in the deterministic framework for the stochastic parabolic problem is the issue of positivity of the density. We solve this problem by using a regularizing effect of parabolic equations with drifts and a bound given by the entropy, quite in the spirit of Mellet, Vasseur, [59], cf. Theorem A.1. Then, the proof of convergence of the parabolic approximation (3.1) to Problem (1.1) is adapted from the proof in the deterministic case to circumvent two additional difficulties:

1. there is a lack of compactness with respect to ω; one has to pass to the limit in some stochastic integrals,
2. there are no "uniform in ε" L^∞ bounds on solutions (here ε is the regularization parameter in the parabolic problem (3.1)).

Problem 1. is solved by use of convergence in law and martingale formulations, Problem 2. is solved by using higher moment estimates (see (3.10) and (3.11)–(3.12)). We will give more details about the main problematic of the paper in Section 2.4, after our framework has been introduced more precisely. Note that Problem 2. also occurs in the resolution of the isentropic Euler system for flows in non-trivial geometry, as treated by Le Floch, Westdickenberg, [53].

2. Notations and main result

2.1. Stochastic forcing

Our hypotheses on the stochastic forcing term $\Phi(\rho,u)W(t)$ are the following ones. We assume that $W = \sum_{k\geq 1}\beta_k e_k$ where the β_k are independent Brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal system in a Hilbert space $\mathfrak{U}$. For each $\rho \geq 0$, $u \in \mathbb{R}$, $\Phi(\rho,u)\colon \mathfrak{U} \to L^2(\mathbb{T})$ is defined by

$$\Phi(\rho,u)e_k = \sigma_k(\cdot,\rho,u) = \rho\sigma_k^*(\cdot,\rho,u), \tag{2.1}$$

where $\sigma_k^*(\cdot,\rho,u)$ is a 1-periodic continuous function on $\mathbb{R}$. More precisely, we assume $\sigma_k^* \in C(\mathbb{T}_x \times \mathbb{R}_+ \times \mathbb{R})$ and the bound

$$\mathbf{G}(x,\rho,u) := \left(\sum_{k\geq 1} |\sigma_k(x,\rho,u)|^2\right)^{1/2} \leq A_0\rho\left[1+u^2+\rho^{2\theta}\right]^{1/2}, \tag{2.2}$$

for all $x \in \mathbb{T}$, $\rho \geq 0$, $u \in \mathbb{R}$, where A_0 is some non-negative constant. Depending on the statement, we will sometimes also make the following localization hypothesis: for $\varkappa > 0$, denote by $z = u - \rho^\theta$, $w = u + \rho^\theta$ the Riemann invariants for (1.1) and by $\Lambda_\varkappa$ the domain

$$\Lambda_\varkappa = \{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R}; -\varkappa \leq z \leq w \leq \varkappa\}. \tag{2.3}$$

We will establish some of our results (more precisely: the resolution of the approximate parabolic Problem (3.1)) under the hypothesis that there exists $\varkappa > 0$ such that

$$\operatorname{supp}(\mathbf{G}) \subset \mathbb{T}_x \times \Lambda_\varkappa. \tag{2.4}$$

We define the auxiliary space $\mathfrak{U}_0 \subset \mathfrak{U}$ by

$$\mathfrak{U}_0 = \left\{v = \sum_{k\geq 1}\alpha_k e_k;\ \sum_{k\geq 1}\frac{\alpha_k^2}{k^2} < \infty\right\}, \tag{2.5}$$