ISSN 0012-9593

quatrième série - tome 52 fascicule 4 juillet-août 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Sang-hyun KIM & Thomas KOBERDA & Yash LODHA

Chain groups of homeomorphisms of the interval

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 52, 2019, p. 797 à 820

CHAIN GROUPS OF HOMEOMORPHISMS OF THE INTERVAL

BY SANG-HYUN KIM, THOMAS KOBERDA AND YASH LODHA

ABSTRACT. – We introduce and study the notion of a chain group of homeomorphisms of a one-manifold, which is a certain generalization of Thompson's group F. The resulting class of groups exhibits a combination of uniformity and diversity. On the one hand, a chain group either has a simple commutator subgroup or the action of the group has a wandering interval. In the latter case, the chain group admits a canonical quotient which is also a chain group, and which has a simple commutator subgroup. On the other hand, every finitely generated subgroup of $\mathrm{Homeo}^+(I)$ can be realized as a subgroup of a chain group. As a corollary, we show that there are uncountably many isomorphism types of chain groups, as well as uncountably many isomorphism types of countable simple subgroups of $\mathrm{Homeo}^+(I)$. We consider the restrictions on chain groups imposed by actions of various regularities, and show that there are uncountably many isomorphism types of 3-chain groups which cannot be realized by C^2 diffeomorphisms, as well as uncountably many isomorphism types of 6-chain groups which cannot be realized by C^1 diffeomorphisms. As a corollary, we obtain uncountably many isomorphism types of simple subgroups of $\mathrm{Homeo}^+(I)$ which admit no nontrivial C^1 actions on the interval. Finally, we show that if a chain group acts minimally on the interval, then it does so uniquely up to topological conjugacy.

RÉSUMÉ. – Nous introduisons la notion d'un groupe de chaînes d'homéomorphismes d'une variété de dimension un, ce qui est une certaine généralisation du groupe F de Thompson. La classe des groupes qui en résulte profite de quelques phénomènes d'uniformité et de diversité. D'un côté, un groupe de chaînes possède un sous-groupe de commutateur simple, sinon l'action du groupe possède un intervalle d'errance. Dans ce dernier cas, le groupe de chaînes admet un quotient canonique qui est aussi un groupe de chaînes dont le sous-groupe de commutateur est simple. D'autre part, chaque sous-groupe engendré d'un sous-ensemble fini de $\mathrm{Homeo}^+(I)$ peut être réalisé comme sous-groupe d'un groupe de chaînes. Il en résulte que les classes d'isomorphisme des groupes de chaînes sont indénombrables, ainsi que les classes d'isomorphisme des sous-groupes simples dénombrables de $\mathrm{Homeo}^+(I)$ sont indénombrables. En outre, nous considérons les restrictions imposées sur les groupes de chaînes par la régularité, et nous démontrons l'existence de nombreux groupes de 3-chaînes qui n'admettent aucune action fidèle de classe C^2 sur une variété de dimension un, et de nombreux groupes de 6-chaînes qui n'admettent aucune action de classe C^1 sur une variété de dimension un. Il en résulte que les classes d'isomorphisme des sous-groupes simples dénombrables de $\mathrm{Homeo}^+(I)$ qui n'agissent pas d'une manière non triviale sur l'intervalle sont indénombrables. Enfin, nous démontrons qu'un groupe

 doi:10.24033/asens.2397

de chaînes qui agit sur l'intervalle d'une manière minimale agit d'une manière unique, à conjugué topologique près.

1. Introduction

In this paper, we introduce and study the notion of a chain group of homeomorphism of a connected one-manifold. A chain group can be viewed as a generalization of Thompson's group F which sits inside the group of homeomorphisms of the manifold in a particularly nice way.

We denote by $\mathrm{Homeo}^+(\mathbb{R})$ the group of orientation preserving homeomorphisms on $\mathbb{R}$. The *support* of $f \in \mathrm{Homeo}^+(\mathbb{R})$ is the set of $x \in \mathbb{R}$ such that $f(x) \neq x$. The support of a group $G \leq \mathrm{Homeo}^+(\mathbb{R})$ is defined as the union of the supports of all the elements in G. For an interval $J \subseteq \mathbb{R}$, let us denote the left- and the right-endpoints of J by $\partial^- J$ and $\partial^+ J$, respectively.

Suppose $\mathscr{J} = \{J_1, \ldots, J_n\}$ is a collection of nonempty open subintervals of $\mathbb{R}$. We call $\mathscr{J}$ a *chain of intervals* (or an *n-chain of intervals* if the cardinality of $\mathscr{J}$ is important) if $J_i \cap J_k = \emptyset$ if $|i - k| > 1$, and if $J_i \cap J_{i+1}$ is a proper nonempty subinterval of J_i and J_{i+1} for $1 \leq i \leq n - 1$. See Figure 1.

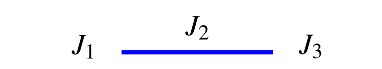

FIGURE 1. A chain of three intervals.

SETTING 1.1. – *We let $n \geq 2$ and let $\mathscr{J} = \{J_1, \ldots, J_n\}$ be a chain of intervals such that $\partial^- J_i < \partial^- J_{i+1}$ for each $i < n$. We consider a collection of homeomorphisms $\mathscr{F} = \{f_1, \ldots, f_n\}$ such that* $\operatorname{supp} f_i = J_i$ *and such that $f_i(t) \geq t$ for each $t \in \mathbb{R}$. We set $G_{\mathscr{F}} = \langle \mathscr{F} \rangle \leq$* $\mathrm{Homeo}^+(\mathbb{R})$.

We call the group $G = G_{\mathscr{F}}$ a *prechain group*. We say that G is a *chain group* (sometimes *n-chain group*) if, moreover, the group $\langle f_i, f_{i+1} \rangle$ is isomorphic to Thompson's group F for each $i = 1, 2, \ldots, n - 1$. Whereas this definition may seem rather unmotivated, part of the purpose of this article is to convince the reader that chain groups are natural objects.

In this paper, we will consider chain groups as both abstract groups and as groups with a distinguished finite generating set $\mathscr{F}$ as above. Whenever we mention "the generators" of a chain group, we always mean the distinguished generating set $\mathscr{F}$ which realizes the group as a chain group of homeomorphisms.

1.1. Main results

Elements of the class of chain groups enjoy many properties which are mostly independent of the choices of the homeomorphisms generating them, and at the same time can be very diverse. Moreover, chain groups are abundant in one-dimensional dynamics. Our first result establishes the naturality of chain groups:

THEOREM 1.1. – *If G is a prechain group as in Setting 1.1, then the group*

$$G_N := \langle f^N \mid f \in \mathscr{F} \rangle \leq \mathrm{Homeo}^+(\mathbb{R})$$

is a chain group for all sufficiently large N.

In Theorem 1.1 and throughout this paper, N a sufficiently large exponent means that N is larger than some natural number which depends on the particular generators of the chain group.

Choosing a two-chain group whose generators are C^∞ diffeomorphisms of $\mathbb{R}$, we obtain the following immediate corollary, which is a complement to (and at least a partial recovery of) a result of Ghys-Sergiescu [17]:

COROLLARY 1.2. – *Thompson's group F can be realized as a subgroup of* $\mathrm{Diff}^\infty(I)$*, the group of C^∞ orientation preserving diffeomorphisms of the interval.*

An action of a group G on a topological space X is *minimal* if every orbit of G is dense. We write $\operatorname{supp} G = \bigcup_{g \in G} \operatorname{supp} g$. We will say a chain group G is *minimal* if its action on $\operatorname{supp} G$ is minimal. General chain groups have remarkably uncomplicated normal subgroup structure, in relation to their dynamical features:

THEOREM 1.3. – *For an n-chain group G, exactly one of the following holds:*

(i) *The action of G is minimal; in this case, every proper quotient of G is abelian and the commutator subgroup $G' \leq G$ is simple;*

(ii) *The closure of some G-orbit is perfect and totally disconnected; in this case, G canonically surjects onto a minimal n-chain group.*

A general chain group may fail to have a simple commutator subgroup (Proposition 4.8).

Every finitely generated subgroup of $\mathrm{Homeo}^+(\mathbb{R})$ embeds into a minimal chain group, so that the subgroup structure of chain groups can be extremely complicated:

THEOREM 1.4. – *Let $G = \langle f_1, f_2, \ldots, f_n \rangle \leq \mathrm{Homeo}^+(\mathbb{R})$ for some $n \geq 2$.*

1. *Then G embeds into an $(n+2)$-chain group.*
2. *If* $\operatorname{supp} f_1$ *has finitely many components, then G embeds into an $(n+1)$-chain group.*

The notion of *rank* of a chain group is somewhat subtle. Indeed, the next proposition shows that a given n-chain group not only contains m-chain groups for all m, but is in fact isomorphic to an m-chain group for all $m \geq n$:

PROPOSITION 1.5. – *For $m \geq n \geq 2$, each n-chain group is isomorphic to some m-chain group.*