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SMOOTH TIMES OF A FLOW IN DIMENSION 1

BY Hélène EYNARD-BONTEMPS

ABSTRACT. – Let α be an irrational number and I an interval of $\mathbb{R}$. If α is *Diophantine*, we show that any one-parameter group of homeomorphisms of I whose time-1 and time-α maps are C^∞ is in fact the flow of a C^∞ vector field. If α is *Liouville* on the other hand, we construct a one-parameter group of homeomorphisms of I whose time-1 and time-α maps are C^∞ but which is not the flow of a C^2 vector field (though, if I has boundary, we explain that the hypotheses force it to be the flow of a C^1 vector field). We extend both results to *families* of irrational numbers, the critical arithmetic condition in this case being *simultaneous "diophantinity"*.

For one-parameter groups defining a *free* action of $(\mathbb{R}, +)$ on I, these results follow from famous linearization theorems for circle diffeomorphisms. The novelty of this work concerns non-free actions.

RÉSUMÉ. – Soit α un nombre irrationnel et I un intervalle de $\mathbb{R}$. Si α est *diophantien*, on montre que tout groupe à un paramètre d'homéomorphismes de I dont les temps 1 et α sont de classe C^∞ est en fait le flot d'un champ de vecteurs C^∞. Si au contraire α est *de Liouville*, on construit un groupe à un paramètre d'homéomorphismes de I dont les temps 1 et α sont de classe C^∞ mais qui n'est pas le flot d'un champ de vecteurs C^2 (toutefois, si I a un bord non vide, on explique qu'il s'agit automatiquement du flot d'un champ C^1). On étend ces deux résultats à des familles de nombres irrationnels, la condition arithmétique critique étant dans ce cas le caractère « simultanément diophantien ».

Pour des groupes à un paramètre définissant une action *libre* de $(\mathbb{R}, +)$ sur I, ces résultats découlent de célèbres théorèmes de linéarisation pour les difféomorphismes du cercle. La nouveauté de ce travail concerne les actions non libres.

I wish to dedicate this article to the memory of Jean-Christophe Yoccoz, who suggested to me, as I was still a PhD student, a "program" which led to the first "half" (Section 2) of this article many years later. He was right, it was "tractable". I also wish to thank Christian Bonatti, Sylvain Crovisier and Bassam Fayad for everything they taught me about one-dimensional dynamics (and Bassam for his precious advice on this particular subject), as well as the organizers of the "Aussois winter school of geometry and dynamics," whose invitation motivated me to dive back into this long-term project.

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1. Introduction

Standing assumptions and vocabulary. – In this article, all vector fields are assumed time-independent, of regularity at least C^1, and complete. By the *flow* of such a vector field on an interval I, we mean the one-parameter group of diffeomorphisms of I made of its time-t maps, for t in $\mathbb{R}$. We will sometimes refer to a one-parameter group of homeomorphisms of I as a "C^0 flow" on I, and sometimes confuse such a group with the continuous $\mathbb{R}$-action that it defines on I, thus talking about the "time-t maps of the action".

1.1. Motivation and results

This work is initially motivated by the study of smooth $\mathbb{Z}^2$-actions on the segment and their possible deformations, in relation with codimension-one foliations of 3-manifolds (cf. [5]). In order to manipulate such actions, one can try to describe the *centralizer* of a given element of the group of smooth diffeomorphisms of the segment. To that end, one first needs to understand the local picture near an isolated fixed point. This motivates the study of (the centralizer of) C^∞-diffeomorphisms f of $\mathbb{R}_+$ without fixed points in $\mathbb{R}_+^* = (0, +\infty)$. These divide into *contractions* and *expansions* satisfying $f(x) < x$ (resp. $f(x) > x$) for every $x \in \mathbb{R}_+^*$. Since it is lighter to mention "contractions" rather than "diffeomorphisms of $\mathbb{R}_+$ without fixed points in $\mathbb{R}_+^*$," we will focus on contractions in the next paragraphs (except in the statements 1.1 through 1.3 which we kept as general as possible), but everything works for expansions as well. Of course, $\mathbb{R}_+$ can be replaced by any semi-open interval.

One can obtain a contraction by taking the time-1 map of a *smooth contracting* vector field on $\mathbb{R}_+$, that is a vector field vanishing only at 0 and "pointing leftwards" everywhere else, i.e., of the form $u\partial_x$ (where x is the coordinate on $\mathbb{R}_+$) with $u : \mathbb{R}_+ \to \mathbb{R}_-$ vanishing only at 0 (we will often identify the vector field with the corresponding function u).

And one actually has the following *partial* converse:

THEOREM 1.1 (Szekeres [16], Sergeraert [15], Yoccoz [19]). – *Let $k \in \mathbb{N}$, $k \geq 2$, and let f be a C^k-diffeomorphism of $\mathbb{R}_+$ without fixed points in $\mathbb{R}_+^*$. Then f is the time-*1 *map of the flow of a complete vector field of class C^1 on $\mathbb{R}_+$ and C^{k-1} on $\mathbb{R}_+^*$.*

One cannot hope for more than C^1 regularity on $\mathbb{R}_+$ in general in the above statement, as Sergeraert shows in [15] by exhibiting a C^∞ contraction f which does not imbed in any C^2 flow (cf. Section 3.1.1 for an outline of his construction). This fact is of importance to us because of the following:

THEOREM 1.2 ("Kopell's Lemma" [13]). – *Let f and g be two commuting diffeomorphisms of $\mathbb{R}_+$ of class C^2 and C^1 respectively. If f has no fixed point in $\mathbb{R}_+^*$ and g has one, then $g =$ *id.

COROLLARY 1.3 (cf. for example [14]). – *Let $k \in \mathbb{N}$, $k \geq 2$, and let f be a C^k-diffeomorphism of $\mathbb{R}_+$ without fixed points in $\mathbb{R}_+^*$. Then f is the time-*1 *map of the flow of a* unique *C^1 vector field on $\mathbb{R}_+$, which we call the* Szekeres vector field of f. *This vector field is C^{k-1} on $\mathbb{R}_+^*$, and the C^1-centralizer of f coincides with its flow.*

In particular, for a C^∞ contraction f of $\mathbb{R}_+$, the C^1-centralizer of f, i.e., the set of C^1 diffeomorphisms of $\mathbb{R}_+$ commuting with f, consists in a one-parameter group of C^1-diffeomorphisms which are actually C^∞ when restricted to the open half-line. Now the C^∞-centralizer consists precisely of those flow maps of the Szekeres vector field ξ which are smooth *on all of* $\mathbb{R}_+$. Let us denote by $\mathcal{S}_\xi$ the subgroup of $\mathbb{R}$ made of the times t for which the time-t map of ξ is smooth. This subgroup contains $\mathbb{Z}$ in the present situation since the time-1 map f is assumed smooth. It can be all of $\mathbb{R}$, when ξ itself is smooth, in which case the centralizer of f is a one-parameter group of diffeomorphisms (in particular path-connected). But it can also be reduced to $\mathbb{Z}$, in which case the centralizer of f is infinite cyclic, generated by f, which is the case in Sergeraert's construction mentioned above.

In order to study $\mathbb{Z}^n$-actions on one-dimensional manifolds and their possible deformations, it is important to know whether $\mathcal{S}_\xi$ can be neither connected nor infinite cyclic (cf. [2]). The author answered this question in [4], combining Sergeraert's construction with Anosov-Katok-like methods of deformation by conjugation (introduced in [1]; see also [7] and the references therein) to construct a contracting vector field whose time-1 and time-α maps are smooth, for some irrational number α, but whose time-$\frac{1}{2}$ map is not C^2. Hence, the set of smooth times is dense in $\mathbb{R}$ (it actually contains a Cantor set), but is not all of $\mathbb{R}$.

In the construction of [4], the very good approximation of α by rational numbers played a crucial role. To make this statement more precise, let us recall the famous partition of $\mathbb{R} \setminus \mathbb{Q}$ into Diophantine and Liouville numbers. A real number α is said to satisfy a Diophantine condition of order $\nu > 0$ if there exists a constant $C > 0$ such that for every $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$, $|q\alpha - p| > \frac{C}{q^{1+\nu}}$, or in other words such that for every $q \in \mathbb{N}^*$, $\|q\alpha\| > \frac{C}{q^{1+\nu}}$, where $\|q\alpha\|$ denotes the distance between $q\alpha$ and $\mathbb{Z}$. A *Diophantine number* is a number satisfying such a condition for some $\nu > 0$ (in particular, such a number is necessarily irrational), and a *Liouville number* is an irrational number which is not Diophantine. Roughly speaking, Diophantine and Liouville numbers are respectively "badly" and "well" approximated by rational numbers.

For the construction of [4] to work, α needed to be Liouville, and the author proved shortly afterwards in the (unsubmitted) preprint [6] that one could actually make the construction work *for any* Liouville number α (cf. Theorem A, case $d = 1$). It was then natural to wonder whether, conversely, the presence, along with 1, of a Diophantine number α in the set of smooth times of a C^1 contracting vector field would force the latter to be C^∞ itself (cf. below for "evidence" pointing in this direction). As it turns out, the "contracting" hypothesis plays no role at that point, and the main new achievement of this paper is to give a positive answer to the last question for *any* C^1 vector field on any interval (cf. Theorem B, case $d = 1$).

These two statements (for α Liouville and α Diophantine respectively) correspond to the case "$d = 1$" in Theorems A and B below, which extend them to *families* of irrational numbers, for which we have the following "family-version" of the dichotomy Diophantine/Liouville: we say that some numbers $\alpha_1, \dots, \alpha_d$, with $d \in \mathbb{N}^*$, are *simultaneously Diophantine* if there exist $\nu > 0$ and $C > 0$ such that for every $q \in \mathbb{N}^*$,