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LÉVY-KHINTCHIN THEOREM FOR BEST SIMULTANEOUS DIOPHANTINE APPROXIMATIONS

BY Yitwah CHEUNG AND Nicolas CHEVALLIER

ABSTRACT. – We extend two results about the ordinary continued fraction expansion to best simultaneous Diophantine approximations of vectors or matrices. The first result is the Lévy-Khintchin theorem about the almost sure growth rate of the denominators of the convergents. The second result is a theorem of Doeblin and of Bosma, Jager and Wiedijk about the almost sure limit distribution of the sequence of products $q_n \, \mathrm{d}(q_n\theta, \mathbb{Z})$ where the q_n's are the denominators of the convergents associated with the real number θ by the ordinary continued fraction algorithm. Besides these two main results, we show that when $d \geq 2$, for almost all vectors $\theta \in \mathbb{R}^d$, $\liminf_{n\to\infty} q_{n+k} \, \mathrm{d}(q_n\theta, \mathbb{Z}^d)^d = 0$ for all positive integers k, where $(q_n)_{n\in\mathbb{N}}$ is the sequence of best approximation denominators of θ.

RÉSUMÉ. – Nous étendons deux résultats sur le développement en fraction continue ordinaire aux meilleures approximations diophantiennes simultanées de vecteurs ou de matrices. Le premier résultat est le théorème de Lévy-Kintchin sur le taux de croissance presque sûr des dénominateurs des réduites. Le second est un théorème de Doeblin et de Bosma, Jager et Wiedijk sur la distribution presque sûre de la suite des produits $q_n \, \mathrm{d}(q_n\theta, \mathbb{Z})$ où les q_n sont les dénominateurs des réduites associées au nombre réel θ par l'algorithme des fractions continues ordinaire. En dehors de ces deux résultats principaux, nous montrons que lorsque $d \geq 2$, pour presque tous les vecteurs $\theta \in \mathbb{R}^d$, $\liminf_{n\to\infty} q_{n+k} \, \mathrm{d}(q_n\theta, \mathbb{Z}^d)^d = 0$ pour tous les entiers positifs k, où $(q_n)_{n\in\mathbb{N}}$ est la suite des dénominateurs des meilleures approximations de θ.

1. Introduction

In 1936, Aleksandr Khintchin showed that there exists a constant γ such that the denominators $(q_n)_{n\geq 0}$ of the convergents of the continued fraction expansions of almost all real numbers θ satisfy

$$\lim_{n\to\infty} q_n^{1/n} = \gamma$$

(see [28]). Soon afterward, in [34], in the footnote page 289, Paul Lévy gave the explicit value of the constant,

$$\gamma = \exp \frac{\pi^2}{12 \ln 2}.$$

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In [17], published in 1940, among many other results about the ordinary continued fraction expansion, Wolfgang Doeblin stated the following result: for almost all real numbers θ,

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{card}\{0 \le k < n : q_k \, \mathrm{d}(q_k\theta, \mathbb{Z}) \le t\} = g(t)$$

for all $t \in [0, 1]$, where

$$g(t) = \int_0^t \frac{1}{2\ln 2} \frac{1 - |1-2s|}{s} \, ds.$$

Doeblin only sketched the proof of this result and it is difficult to reconstitute a complete proof from his paper. In [26], Iosifescu gave a very interesting account of Doeblin's paper. Doeblin's outstanding article was not immediately noticed, so the above result was first called the Lenstra conjecture, now it is often called the Doeblin-Lenstra conjecture. A complete proof of the Doeblin-Lenstra conjecture was given in 1983 by Wieb Bosma, Hendrik Jager and Freek Wiedijk, see [5]. Later, Jager proved variants of this result, in particular with the quantity $q_{k+1} \, \mathrm{d}(q_k\theta, \mathbb{Z})$ instead of $q_k \, \mathrm{d}(q_k\theta, \mathbb{Z})$ (see [27] for more results).

The aim of our work is to extend to the best simultaneous Diophantine approximations, both the Lévy-Khintchin result and the Jager version of the Doeblin, Bosma, Jager and Wiedijk result. We choose the Jager version because one of the striking difference between one-dimensional and multidimensional Diophantine approximations is that the classic one-dimensional lower bound $q_{k+1} \, \mathrm{d}(q_k\theta, \mathbb{Z}) \ge 1/2$ no longer holds in the multidimensional setting; this explains for example that the only singular vectors are the rational numbers in dimension one.

Let d and c be two positive integers. Suppose $\mathbb{R}^d$ and $\mathbb{R}^c$ are endowed with the standard Euclidean norms $\|.\|_{\mathbb{R}^d}$ and $\|.\|_{\mathbb{R}^c}$. We prove

THEOREM 1. – *There exists a constant $L_{d,c}$ such that for almost all matrices $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|Q_n(\theta)\|_{\mathbb{R}^c} = L_{d,c} \,,$$

$$\lim_{n\to\infty} \frac{1}{n} \ln(\mathrm{d}(\theta Q_n(\theta), \mathbb{Z}^d)) = -\frac{c}{d} L_{d,c},$$

where $Q_n(\theta) \in \mathbb{Z}^c$, $n \ge 0$, is the sequence of best Diophantine approximation denominators of θ associated with the norms $\|.\|_{\mathbb{R}^d}$ and $\|.\|_{\mathbb{R}^c}$.

(See Section 2.1 for the definition of best Diophantine approximation denominators.)

Theorem 1 is stated for standard Euclidean norms but it holds for all pairs of Euclidean norms on $\mathbb{R}^d$ and $\mathbb{R}^c$ with the same constant $L_{d,c}$ (see Theorem 6.1). We are convinced that Theorem 1 is valid for general norms but we do not know whether the constants $L_{d,c}$ depend on the norms (see Section 10).

For a matrix θ in $\mathrm{M}_{d,c}(\mathbb{R})$, let us denote $\beta_n(\theta) = \|Q_{n+1}(\theta)\|_{\mathbb{R}^c}^c \, \mathrm{d}(\theta Q_n(\theta), \mathbb{Z}^d)^d$ and for a real number a, let us denote δ_a the Dirac measure at a.

THEOREM 2. – 1. *There exists a probability measure $\nu_{d,c}$ on $\mathbb{R}$ such that, for almost all matrices $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$, $\nu_{d,c}$ is the weak limit of the sequence of probability measures*

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{\beta_k(\theta)}.$$

2. *The support of the measure* $\nu_{d,c}$ *is included in a bounded interval, and contains* 0 *provided that* $c + d \geq 3$.

The Lévy-Khintchin result has already been extended to multidimensional settings. For instance, for almost all θ in $\mathbb{R}^d$, the denominators $(J_n(\theta))_{n\geq 0}$ of the Jacobi-Perron expansion of θ satisfy $\lim_{n\to\infty} \frac{1}{n} \ln J_n(\theta) = c_d$ for some constant c_d (see [6]). The common proofs of such results use ergodic theory. The one-dimensional Lévy-Khintchin result can be proven with the Birkhoff ergodic theorem, while the growth rate of the Jacobi-Perron denominators can be derived from the Oseledec multiplicative ergodic theorem. In both cases, the proof depends on the existence of an underlying dynamical system: the Gauss map or the Jacobi-Perron map (see [47] for many examples of these kinds of maps). However, no such map associated with best Diophantine approximations is known when $d + c \geq 3$. One way to circumvent this problem is to use the left action of the diagonal flow

$$g_t = \begin{pmatrix} e^{ct} I_d & 0 \\ 0 & e^{-dt} I_c \end{pmatrix} \in \mathrm{SL}(d+c,\mathbb{R}),\ t \in \mathbb{R},$$

on the space of unimodular lattices $\mathcal{L}_{d+c} = \mathrm{SL}(d+c,\mathbb{R})/\mathrm{SL}(d+c,\mathbb{Z})$. This idea can be traced back to Hermite [24] and has been used extensively in Diophantine approximation since the work of Dani [15, 16]; see for instance [1, 18, 23]. In the same vein, the parametric geometry of numbers introduced recently by Schmidt and Summerer [45, 46] and complemented by Roy [42], studies the evolution of the successive minima of a lattice under the action of the diagonal flow. This led to many results on Diophantine exponents; see for instance [35, 43].

In [13], the diagonal flow is used to prove that the sequence of best Diophantine approximation denominators of almost all θ in $\mathrm{M}_{d,c}(\mathbb{R})$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|Q_n(\theta)\|_{\mathbb{R}^c} \leq K_{d,c}$$

for some constant $K_{d,c}$. When $c = 1$, it is also possible to derive this inequality from a theorem of W. M. Schmidt (see [12]).

As in the aforementioned works, the diagonal flow (g_t) is the main tool. However, unlike e.g., [13], we introduce an important new ingredient: a co-dimension one submanifold transverse to the flow. It should be noticed that many works on continued fraction algorithms also use transversals, we mention two such works.

Firstly, the transformation induced on a well-chosen sub-interval by an interval exchange transformation T is an interval exchange transformation $\hat{T}$ defined with the same number of intervals. The Rauzy-Veech continued fraction algorithm is the map $R : T \mapsto \hat{T}$ where T is an interval exchange transformation defined on an interval of length 1 and $\hat{T}$ is renormalized so that it is defined on an interval of length 1 (see [38, 39, 50]). In [50], W. Veech proved that R admits a unique absolutely continuous invariant measure up to a scalar multiple, using a bimeasureably invertible extension $\hat{R}$ of the map R. In turn, this extension is constructed as the first return map of a "diagonal flow" on a transversal. It should be noticed that at the same time, H. Masur proposed a very similar construction in [36].

Secondly, A. Haas proved analogs of the result of Doeblin, Bosma, Jager and Wiedijk and of the Lévy-Khintchin result in the setting of excursions of geodesic into the neighborhood