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ON FACTORIZATION AND VECTOR BUNDLES OF CONFORMAL BLOCKS FROM VERTEX ALGEBRAS

BY Chiara DAMIOLINI, Angela GIBNEY AND Nicola TARASCA

Abstract. – Representations of vertex operator algebras define sheaves of coinvariants and conformal blocks on moduli of stable pointed curves. Assuming certain finiteness and semisimplicity conditions, we prove that such sheaves satisfy the factorization conjecture and consequently are vector bundles. Factorization is essential to a recursive formulation of invariants, like ranks and Chern classes, and to produce new constructions of rational conformal field theories and cohomological field theories.

Résumé. – Les représentations des algèbres d'opèrateurs vertex définissent des faisceaux de coinvariants et de blocs conformes sur des modules de courbes pointées stables. En supposant certaines conditions de finitude et de semi-simplicité, nous prouvons que de tels faisceaux satisfont la conjecture de factorisation et sont par conséquent des fibrés vectoriels. La factorisation est essentielle à une formulation récursive des invariants, comme les rangs et les classes de Chern, et à produire de nouvelles constructions de théories conformes rationnelles des champs et de théories cohomologiques des champs.

By assigning a module over a vertex operator algebra to each marked point on a stable pointed curve, one can construct dual vector spaces of coinvariants and conformal blocks, giving rise to sheaves on moduli spaces of stable pointed curves. The main result of this paper is that these sheaves satisfy the factorization property (Theorem 7.0.1), as conjectured in [76, 38]. Namely, if certain finiteness and semisimplicity conditions hold, vector spaces of coinvariants and conformal blocks at a nodal curve decompose as products of analogous spaces at each component of its normalization.

We also show that sheaves of coinvariants satisfy the sewing property (Theorem 8.5.1), a refined version of factorization at infinitesimal smoothings of nodal curves. From this and their projectively flat connection on families of smooth curves [21], we deduce they are vector bundles (VB corollary).

Our findings generalize a number of results known in special cases and for low genus. A historical account with references is given in §§0.2 and 0.3.

Factorization leads to recursive formulas for ranks and is used to show that the Chern characters define semisimple cohomological field theories, hence the Chern classes lie in the

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tautological ring ([22], building on results for affine Lie algebras from [62, 63]). A study of these tautological classes may lead to progress on open questions: As proposed by Pandharipande [69], a computation of such Chern classes independently of the projective flatness of the connection [21] would yield relations in the tautological ring, and could be used to test Pixton's conjecture [70, 52].

For sheaves defined by integrable modules over affine Lie algebras, vector spaces of conformal blocks are canonically isomorphic to generalized theta functions (see [55] and references therein). When in addition the genus is zero, vector bundles of such coinvariants are globally generated [36], hence their Chern classes have positivity properties. For instance, first Chern classes are base-point-free and thus give rise to morphisms, some with images having modular interpretations [45, 46]. It is natural to expect that the more general vector bundles of coinvariants and conformal blocks studied here can be shown to have analogous properties under appropriate assumptions. This has been supported by a preliminary investigation [20].

To outline our results, we set some notation. We refer to a stable pointed coordinatized curve as a triple $(C, P_\bullet, t_\bullet)$ where $(C, P_\bullet)$ is a stable n-pointed curve, $P_\bullet = (P_1, \dots, P_n)$, and $t_\bullet = (t_1, \dots, t_n)$ with t_i a formal coordinate at the point P_i. Let $M^\bullet = (M^1, \dots, M^n)$ be an n-tuple of finitely generated admissible modules over a vertex operator algebra V (see §§1.1-1.2). When $C \setminus P_\bullet$ is affine, the vector space of coinvariants $\mathbb{V}(V; M^\bullet)_{(C,P_\bullet,t_\bullet)}$ is defined as the largest quotient of the tensor product $\bigotimes_{i=1}^n M^i$ by the action of a Lie algebra determined by V and $(C, P_\bullet, t_\bullet)$, see §4.2. In general, by adding more marked points, one can reduce to the case when $C \setminus P_\bullet$ is affine, see (30). Before now, two Lie algebras have been used to define coinvariants: Zhu's Lie algebra $\mathfrak{g}_{C\setminus P_\bullet}(V)$ and the (former) chiral Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$. Here, we introduce a new chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$.

Zhu's Lie algebra (§A.1) is defined when the vertex algebra V is quasi-primary generated and $\mathbb{Z}_{\geq 0}$-graded with lowest degree space of dimension one, for either *fixed* smooth curves [76, 3], or for *rational* stable pointed curves *with coordinates* [67]. To show that $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is a Lie algebra, Zhu uses that any *fixed* smooth algebraic curve admits an atlas such that all transition functions are Möbius transformations. Transition functions between charts on families of curves of arbitrary genus are more complicated, and for an arbitrary vertex operator algebra, $\mathfrak{g}_{C\setminus P_\bullet}(V)$ is not well-defined.

The chiral Lie algebra $\mathscr{L}_{C\setminus P_\bullet}(V)$ is available for curves of arbitrary genus and for more general vertex operator algebras V, not necessarily quasi-primary generated. Defined for smooth pointed curves by Frenkel and Ben-Zvi [38, §19.4.14] and shown in [38] to coincide with that studied by Beilinson and Drinfeld [14], the chiral Lie algebra was extended to nodal curves in [21], enabling the construction of coinvariants on such curves.

In §3, we introduce and study a new chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$, whose coinvariants are projectively flat over $\mathcal{M}_{g,n}$, and additionally satisfy factorization under some natural hypotheses. See §3.3 for a description of $\mathcal{L}_{C\setminus P_\bullet}(V)$ on nodal curves.

In [67], Nagatomo and Tsuchiya remark that the coinvariants on rational curves with coordinates using Zhu's Lie algebra are equivalent to those considered by Beilinson and Drinfeld. In Proposition A.2.1, we verify their genus zero statement, and further show that coinvariants from the chiral Lie algebra are isomorphic to those given by $\mathfrak{g}_{C\setminus P_\bullet}(V)$ whenever Zhu's Lie algebra is defined. For instance, when V is quasi-primary generated and one works over a family of curves admitting an atlas where all transition functions are Möbius transformations, both perspectives are equivalent.

To state our main result, let $(C, P_\bullet, t_\bullet)$ be a stable pointed coordinatized curve, as above, with one node Q. Let $\widetilde{C} \to C$ be the normalization, $Q_\bullet := (Q_+, Q_-)$ the pair of preimages of Q in $\widetilde{C}$, and $s_\bullet := (s_+, s_-)$ with $s_\pm$ a formal coordinate at $Q_\pm$. Let $\mathscr{W}$ be a set of representatives of isomorphism classes of simple V-modules, and for $W \in \mathscr{W}$, let W' be its contragredient module (§1.8).

Factorization theorem (Theorem 7.0.1). – *Let V be a rational, C_2-cofinite vertex operator algebra with one-dimensional weight zero space, and let $M^\bullet$ be an n-tuple of finitely-generated admissible V-modules. One has:*

$$\mathbb{V}\left(V; M^\bullet\right)_{(C,P_\bullet,t_\bullet)} \cong \bigoplus_{W\in\mathscr{W}} \mathbb{V}\left(V; M^\bullet \otimes W \otimes W'\right)_{(\widetilde{C},P_\bullet\sqcup Q_\bullet,t_\bullet\sqcup s_\bullet)}. \tag{1}$$

If $\widetilde{C} = C_+ \sqcup C_-$ is disconnected, with $Q_\pm \in C_\pm$, then (1) is isomorphic to

$$\bigoplus_{W\in\mathscr{W}} \mathbb{V}\left(V; M_+^\bullet \otimes W\right)_{X_+} \otimes \mathbb{V}\left(V; M_-^\bullet \otimes W'\right)_{X_-},$$

where $X_\pm := (C_\pm, P_\bullet|_{C_\pm} \sqcup Q_\pm, t_\bullet|_{C_\pm} \sqcup s_\pm)$, and $M_\pm^\bullet$ are the modules at those points $P_\bullet$ that are in $C_\pm$.

The isomorphism giving the factorization theorem is constructed in §7.

Propositions 3.3.1, 5.1.1, and 6.2.1 are at the heart of this work, and arise from the construction in §2 of the sheaf $\mathcal{V}_C$ globalizing a vertex algebra V over a nodal curve C, integral to the definition of the chiral Lie algebra. The sheaf $\mathcal{V}_C$ is a slight variant on the sheaf $\mathscr{V}_C$ that we defined in [21]. The two sheaves coincide for smooth curves, and as we explain here, we recover the results of [21] using $\mathcal{V}_C$ in place of $\mathscr{V}_C$. So while the projectively flat connection can be obtained in both constructions (see Proposition 2.7.3 for $\mathcal{V}_C$), to have factorization, we have used $\mathcal{V}_C$ (see §2.5, and §2.6). In Proposition 3.3.1, we explicitly describe the chiral Lie algebra on a nodal curve in terms of elements of the chiral Lie algebra on its normalization. This involves stable k-differentials that satisfy an infinite number of identities.

In Proposition 5.1.1, we show that vector spaces of coinvariants defined from the chiral Lie algebra and smooth curves of arbitrary genus are finite-dimensional. Known to be true in special cases, this result is a natural generalization of work of Abe and Nagatomo [3] for coinvariants defined from Zhu's Lie algebra and smooth curves with formal coordinates (see §0.3 for the history of the problem). As in [3], we assume here that V is C_2-cofinite, which implies C_k-cofiniteness for $k \geq 1$ [17, 54].

Proposition 6.2.1 reinterprets coinvariants at a nodal curve as coinvariants on the normalization by the action of a Lie subalgebra of the chiral Lie algebra. The proof of this result, for which we assume V is C_1-cofinite, uses that lowest weight V-modules admit spanning sets of PBW-type, following [54, Cor. 3.12].

In §8.7, following Tsuchimoto [72] in the case of simple affine vertex algebras, we show the sheaf of coinvariants $\mathbb{V}(V; M^\bullet)$ is coordinate-free and descends to the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g. As an application of the factorization theorem, we show: