ISSN 0012-9593

quatrième série - tome 57 fascicule 2 mars-avril 2024

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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Part I: the dynamically coherent case

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 480 euros.
Abonnement avec supplément papier :
Europe : 675 €. Hors Europe : 759 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 57, 2024, p. 293 à 349

PARTIALLY HYPERBOLIC DIFFEOMORPHISMS HOMOTOPIC TO THE IDENTITY IN DIMENSION 3 PART I: THE DYNAMICALLY COHERENT CASE

BY Thomas BARTHELMÉ, Sergio R. FENLEY, Steven FRANKEL AND Rafael POTRIE

ABSTRACT. – We study 3-dimensional dynamically coherent partially hyperbolic diffeomorphisms that are homotopic to the identity, focusing on the transverse geometry and topology of the center-stable and center-unstable foliations, and the dynamics within their leaves. We find a structural dichotomy for these foliations, which we use to show that every such diffeomorphism on a hyperbolic or Seifert-fibered 3-manifold is leaf-conjugate to the time-one map of a (topological) Anosov flow. This proves a classification conjecture of Hertz-Hertz-Ures in hyperbolic 3-manifolds and in the homotopy class of the identity of Seifert manifolds.

RÉSUMÉ. – Nous étudions, en dimension trois, les difféomorphismes partiellement hyperboliques qui sont dynamiquement cohérents et homotopes à l'identité. Nous nous focalisons sur la géométrie et la topologie de leurs feuilletages centraux-stable et centraux-instable, ainsi que sur leur dynamique dans les feuilles. Nous obtenons ainsi que la structure de ces feuilletages doit satisfaire à une dichotomie. En utilisant cette dichotomie, nous montrons que, lorsque la 3-variété est hyperbolique ou de Seifert, les difféomorphismes étudiés ont leur feuilletage central conjugué à celui du temps un d'un flot d'Anosov topologique. Ceci prouve une conjecture de Hertz-Hertz-Ures pour les variétés hyperboliques et dans la classe d'homotopie de l'identité pour les variétés de Seifert.

1. Introduction

A diffeomorphism f of a 3-manifold M is *partially hyperbolic* if it preserves a splitting of the tangent bundle TM into three 1-dimensional sub-bundles

$$TM = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}},$$

T. Barthelmé was partially supported by the NSERC (Funding reference number RGPIN-2017-04592). S. Fenley was partially supported by Simons Foundation grant number 280429. S. Frankel was partially supported by National Science Foundation grant number DMS-1611768. R. Potrie was partially supported by CSIC 618 and ANII-FCE–135352.

 doi:10.24033/asens.2575

where the *stable bundle* E^{s} is eventually contracted, the *unstable bundle* E^{u} is eventually expanded, and the *center bundle* E^{c} is distorted less than the stable and unstable bundles at each point.

From a dynamical perspective, the interest in partial hyperbolicity stems from its appearance as a generic consequence of certain dynamical conditions, such as stable ergodicity and robust transitivity. For example, a diffeomorphism is *transitive* if it has a dense orbit, and *robustly transitive* if this behavior persists under C^1-small deformations. Every robustly transitive diffeomorphism on a 3-manifold is either Anosov or "weakly" partially hyperbolic [30]. Analogous results hold for stable ergodicity and in higher dimensions [11].

From a geometric perspective, one can think of partial hyperbolicity as a generalization of the discrete behavior of Anosov flows, which feature prominently in the theory of 3-manifolds. Recall that a flow Φ on a 3-manifold M is *Anosov* if it preserves a splitting of the unit tangent bundle TM into three 1-dimensional sub-bundles

$$TM = E^{\mathrm{s}} \oplus T\Phi \oplus E^{\mathrm{u}},$$

where $T\Phi$ is the tangent direction to the flow, E^{s} is eventually exponentially contracted, and E^{u} is eventually exponentially expanded. After flowing for a fixed time, an Anosov flow generates a partially hyperbolic diffeomorphism of a particularly simple type, where the stable and unstable bundles are contracted uniformly, and the center direction, which corresponds to $T\Phi$, is left undistorted. More generally, one can construct partially hyperbolic diffeomorphisms of the form $f(x) = \Phi_{\tau(x)}(x)$ where Φ is a (topological) Anosov flow and $\tau\colon M \to \mathbb{R}_{>0}$ is a positive continuous function; the partially hyperbolic diffeomorphisms obtained in this way are called *discretized Anosov flows* (1).

In this article and its sequel [9] we show that large classes of partially hyperbolic diffeomorphisms can be identified with discretized Anosov flows. This confirms a large part of the well-known conjecture by Pujals [16], which attempts to classify 3-dimensional partially hyperbolic diffeomorphisms by asserting that they are all either discretized Anosov flows or deformations of certain kinds of algebraic examples.

1.1. Homotopy, integrability, and conjugacy

There are two important obstructions to identifying a partially hyperbolic diffeomorphism with a discretized Anosov flow. The first comes from the fact that the latter are homotopic to the identity, while the former may be homotopically non-trivial. Homotopically non-trivial examples include Anosov diffeomorphisms on the 3-torus with distinct eigenvalues, "skew products," and the counterexamples to Pujals' conjecture constructed in [15, 13, 17, 12].

The second major obstruction comes from the integrability of the bundles in a partially hyperbolic splitting. In the context of an Anosov flow Φ, the stable and unstable bundles E^{s} and E^{u} integrate uniquely into a pair of 1-dimensional foliations, the *stable* foliation $\mathcal{W}^{\mathrm{s}}$ and *unstable* foliation $\mathcal{W}^{\mathrm{u}}$. In fact, even the *weak stable* and *weak unstable* bundles $E^{\mathrm{s}} \oplus T\Phi$ and $E^{\mathrm{u}} \oplus T\Phi$ integrate uniquely into a transverse pair of Φ-invariant 2-dimensional foliations, the *weak stable* foliation $\mathcal{W}^{ws}$ and *weak unstable* foliation $\mathcal{W}^{wu}$.

(1) Note that a discretized Anosov flow is not in general the time-1 map of a reparametrization of the Anosov flow, see Appendix G.

In the context of a partially hyperbolic diffeomorphism f, the stable and unstable bundles still integrate uniquely into stable and unstable foliations, $\mathcal{W}^s$ and $\mathcal{W}^u$ [42]. However, the *center-stable* and *center-unstable* bundles $E^c \oplus E^s$ and $E^c \oplus E^u$ may fail to be uniquely integrable. In fact, there are examples where it is impossible to find *any* f-invariant 2-dimensional foliation tangent to the center-stable or center-unstable bundle [56, 12].

If one can find a pair of f-invariant foliations tangent to the center-stable and center-unstable bundles then f is said to be *dynamically coherent*. This condition is certainly satisfied if f is a discretized Anosov flow (cf. Appendix G).

We take dynamical coherence as an assumption in the present article; Part II [9] works without this.

1.2. Results

Most of the existing progress towards classifying partially hyperbolic diffeomorphisms takes an outside-in approach, restricting attention to particular classes of manifolds, and comparing to an a priori known model partially hyperbolic (see [28, 42] for recent surveys). In particular, partially hyperbolic diffeomorphisms have been completely classified in manifolds with solvable or virtually solvable fundamental group [40, 41]. Here, by classification we mean both the description of the topology of manifolds and isotopy classes admitting such dynamics as well as the production of topological models that describe such systems.

Ours is an inside-out approach, using the theory of foliations to understand the way the local structure that defines partial hyperbolicity is pieced together into a global picture. We then relate the dynamics of these foliations to the large-scale structure of the ambient manifold. An advantage of this method is that, since it does not rely on a model partially hyperbolic to compare to, we can consider any manifold, not just one where an Anosov flow is known to exist. Note that here it will be possible to construct a topological model even if the manifold is not known to admit an Anosov flow, nor if such a flow is unique.

The following two theorems are the main consequences of our work, applied to two of the major classes of 3-manifolds. Note that the classification of partially hyperbolic diffeomorphisms is always considered up to finite lifts and iterates, since one can easily build infinitely many different but uninteresting examples by taking finite covers.

Theorem A (Seifert manifolds). – *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism on a closed Seifert-fibered* 3*-manifold. If f is homotopic to the identity, then some iterate is a discretized Anosov flow.*

We eliminate the assumption of dynamical coherence in [9]; this resolves the Pujals' Conjecture for Seifered fibered manifolds [2]. Note that Theorem A does not use the classification of Anosov flows on Seifert-fibered 3-manifolds [38, 3].

Theorem B (Hyperbolic manifolds). – *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism on a closed hyperbolic* 3*-manifold. Then some iterate of f is a discretized Anosov flow.*

[2] The conjecture is true for Seifert manifolds with fundamental group with polynomial growth [40] and false in Seifert-fibered manifolds when the isotopy class is not the identity as the examples in [13, 12] are not homotopic to identity and so cannot be discretized Anosov flows.