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ON THE HOMOTOPY THEORY OF STRATIFIED SPACES

BY PETER J. HAINE

ABSTRACT. – Let P be a poset. We define a new homotopy theory of suitably nice P-stratified topological spaces with equivalences on strata and links inverted. We show that the exit-path construction of MacPherson, Treumann, and Lurie defines an equivalence from our homotopy theory of P-stratified topological spaces to the ∞-category of ∞-categories with a conservative functor to P. This proves a stratified form of Grothendieck's homotopy hypothesis, verifying a conjecture of Ayala-Francis-Rozenblyum. Our homotopy theory of stratified spaces has the added benefit of capturing all examples of geometric interest: conically stratified spaces fit into our theory, and the Ayala-Francis-Tanaka-Rozenblyum homotopy theory of conically smooth stratified spaces embeds into ours.

RÉSUMÉ. – Soit P un ensemble ordonné. On définit une nouvelle théorie d'homotopie d'espaces topologiques P-stratifiés agréables en un sens convenable, avec des équivalences sur les strates et des liens inversés. On montre que la construction du « chemin de sortie » de MacPherson, Treumann et Lurie définit une équivalence de notre théorie d'homotopie des espaces topologiques P-stratifiés avec la ∞-catégorie des ∞-catégories avec un foncteur conservatif à P. Cela démontre une forme stratifiée de l'hypothèse d'homotopie de Grothendieck, et prouve une conjecture d'Ayala-Francis-Rozenblyum. Cette théorie de l'homotopie des espaces stratifiés a l'avantage supplémentaire d'englober tous les exemples d'intérêt géométrique: les espaces coniquement stratifiés s'intègrent dans notre théorie, et la théorie de l'homotopie d'Ayala-Francis-Tanaka-Rozenblyum des espaces stratifiés coniquement lisses s'intègre dans la nôtre.

0. Introduction

Let T be a topological space. Classically, there are two approaches to understanding the algebraic topology of T. On the one hand, we can work directly with the genuine topological space T and think about homotopy classes of maps to T, homotopy groups, etc. On the other hand, we can work with a more combinatorial object: the *singular simplicial set* or *fundamental ∞-groupoid* of T. This combinatorial object parametrizes points of T, paths in T, homotopies of paths, etc. Grothendieck's celebrated *homotopy hypothesis*, established

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by Kan and Quillen [29, 30, 31, 32, 33, 44], asserts that these two perspectives are equivalent. The equivalence between the homotopy theories of topological spaces and simplicial sets is at the foundation of modern homotopy theory for a good reason: both perspectives are of great utility. Being able to think about homotopy types as topological spaces gives access to many examples and structures coming from the geometry of manifolds. On the other hand, the combinatorial framework of simplicial sets provides a rich context in which to do algebra in the setting of homotopy theory (e.g., study structured ring spectra).

The purpose of this article is to put the homotopy theory of *stratified* spaces on a similarly good footing. One peculiarity is that for a long time only a robust 'combinatorial' approach to the homotopy theory of stratified spaces existed. The combinatorial approach originates from ideas of MacPherson that came out of studying manifolds with singularities and constructible sheaves. MacPherson observed that a constructible sheaf of vector spaces on a (nice) stratified space T is equivalent to a a representation of the *exit-path* category of T. To explain this, consider the simple example of the unit interval $[0,1]$ with stratification $\{0\} \subset [0,1]$. Let F be a sheaf on $[0,1]$ that is locally constant on $(0,1]$, i.e., constructible with respect to the stratification. Since $|_F (0,1]$ is locally constant, each path between points $s, t \in (0,1]$ defines an isomorphism between stalks $F_s \cong F_t$. Using these isomorphisms and the fact that every open neighborhood of 0 intersects $(0,1]$, one can define a *specalization map* $F_0 \to F_1$ relating the stalks of F at 0 and 1. These three pieces of data completely determine the constructible sheaf F. That is, constructible sheaves on $\{0\} \subset [0,1]$ are representations of the A_2-quiver $\bullet \to \bullet$.

More generally, if T is a topological space with a suitably nice stratification by a poset P, then we can associate to T its *exit-path* ∞*-category* with objects points of T, with 1-morphisms *exit paths* flowing from lower to higher strata (and once they exit a stratum are not allowed to return), with 2-morphisms homotopies of exit-paths respecting stratifications, etc. The adjective 'suitably nice' is quite important here because, while the construction of the fundamental ∞-groupoid makes sense for *any* topological space, if the stratification is not sufficiently nice, then exit paths can fail to suitably compose and this informal description cannot be made to actually define an ∞-category. This is part of an overarching problem: there does not exist yet a homotopy theory of stratified spaces that is simple to define, encapsulates examples from topology, and has excellent formal properties. One of the goals of this paper is to resolve this matter.

Treumann [50], Woolf [52], Lurie [35, Appendix A], and Ayala-Francis-Rozenblyum [3, §1] have all worked to realize MacPherson's exit-path construction under a variety of point-set topological assumptions. In various forms, Ayala-Francis-Rozenblyum [3, Conjectures 0.0.4 & 0.0.8], Barwick, and Woolf have all conjectured that the exit-path construction defines an equivalence of ∞-categories from suitably nice stratified spaces (with stratified homotopies inverted) to ∞-categories with a conservative functor to a poset. The main goal of this paper is to prove this conjecture.

There is already strong evidence for the power of having both topological and combinatorial approaches to stratified homotopy theory on the same footing. On the one hand, Ayala-Francis-Tanaka-Rozenblyum have made extensive use of explicit topological methods to study manifolds with singularities and their factorization homology [2, 3, 4, 5]. On the other hand, in joint work with Barwick and Glasman [9], we used the combinatorial perspective on stratified spaces to associate to each variety an 'exit-path category' for the étale topology. The combinatorial perspective is necessary here because 'differential-topological' constructions of exit-path ∞-categories are not available in positive-characteristic algebraic geometry. This étale exit-path category is a powerful globalization of the absolute Galois group of a field. For example, it gives rise to a new and concrete definition of the étale homotopy type of Artin-Mazur-Friedlander [1, 19]. Moreover, much like how the Neukirch-Uchida theorem shows that the absolute Galois group is a complete invariant of number fields [42, 41, 51], the étale exit-path category is a complete invariant of the varieties that appear in Grothendieck's anabelian conjecture [21].

0.1. The stratified homotopy hypothesis

In order to state our stratified homotopy hypothesis, it is useful to get a better understanding of what kind of data should determine a stratified space. A stratification of a topological space T by the poset $\{0 < 1\}$ is given by a closed subspace $T_0 \subset T$ and its open complement $T_1 := T \smallsetminus T_0$. It is natural to ask: given the topological spaces T_0 and T_1, what extra data do we need to reconstruct the stratified space T (up to stratified homotopy equivalence)? Roughly, the answer is *gluing* information called the *(homotopy) link* between the 0-th and 1-st strata. Introduced by Quinn [45], the link $\mathrm{Link}(T_0, T_1)$ is defined as the space of paths $\gamma\colon [0,1] \to T$ such that $\gamma(0) \in T_0$ and for all $s > 0$ we have $\gamma(s) \in T_1$. Evaluation at 0 and 1 define maps $\mathrm{Link}(T_0, T_1) \to T_0$ and $\mathrm{Link}(T_0, T_1) \to T_1$, respectively. In nice situations, the stratified space T can be recovered as a homotopy pushout of the resulting span

$$T_0 \leftarrow \mathrm{Link}(T_0, T_1) \to T_1$$

[45, §2]. For more general stratifications, the idea is that stratified spaces can be reconstructed from the data of all of their strata and all possible links relating strata.

To state our result, we introduce some notation. Given a poset P, we write $\mathbf{Top}_{/P}$ for the category of P-stratified topological spaces (see Definition 1.2.3). Lurie's exit-path construction defines a functor $\mathrm{Sing}_P\colon \mathbf{Top}_{/P} \to \mathrm{s}\mathbf{Set}_{/P}$ from P-stratified spaces the category of simplicial sets over (the nerve of) P (see § 1.2). Write $\mathbf{Top}^{\mathrm{ex}}_{/P} \subset \mathbf{Top}_{/P}$ for the full subcategory of those P-stratified topological spaces T for which Lurie's exit-path simplicial set $\mathrm{Sing}_P(T)$ is an ∞-category. The ∞-categories that arise in this way have a special property: the fibers of the structure morphism to P are given by the fundamental ∞-groupoids of the strata. In particular, every morphism in each fiber is invertible. Equivalently, the structure morphism is a conservative functor. We write $\mathbf{Str}_P$ for the ∞-category of ∞-categories over (the nerve of) P such that the structure morphism is conservative. We refer to $\mathbf{Str}_P$ as the ∞-category of *abstract P-stratified homotopy types*.

Our main result provides an affirmative answer to a conjecture of Ayala-Francis-Rozenblyum [3, Conjectures 0.0.4 & 0.0.8]:

0.1.1. Theorem (stratified homotopy hypothesis; see Theorem 3.2.4 and Corollary 3.3.3).