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TRILINEAR COMPENSATED COMPACTNESS AND BURNETT'S CONJECTURE IN GENERAL RELATIVITY

BY CÉCILE HUNEAU AND JONATHAN LUK

ABSTRACT. – Consider a sequence of C^4 Lorentzian metrics $\{h_n\}_{n=1}^{+\infty}$ on a manifold $\mathcal{M}$ satisfying the Einstein vacuum equation $\mathrm{Ric}(h_n) = 0$. Suppose there exists a smooth Lorentzian metric h_0 on $\mathcal{M}$ such that $h_n \to h_0$ uniformly on compact sets. Assume also that on any compact set $K \subset \mathcal{M}$, there is a decreasing sequence of positive numbers $\lambda_n \to 0$ such that

$$\|\partial^\alpha (h_n - h_0)\|_{L^\infty(K)} \lesssim \lambda_n^{1-|\alpha|}, \quad |\alpha| \geq 4.$$

It is well-known that h_0, which represents a "high-frequency limit," is not necessarily a solution to the Einstein vacuum equation. Nevertheless, Burnett conjectured that h_0 must be isometric to a solution to the Einstein-massless Vlasov system.

In this paper, we prove Burnett's conjecture assuming that $\{h_n\}_{n=1}^{+\infty}$ and h_0 in addition admit a $\mathbb{U}(1)$ symmetry and obey an elliptic gauge condition. The proof uses microlocal defect measures—we identify an appropriately defined microlocal defect measure to be the Vlasov measure of the limit spacetime. In order to show that this measure indeed obeys the Vlasov equation, we need some special cancellations which rely on the precise structure of the Einstein equations. These cancellations are related to a new "trilinear compensated compactness" phenomenon for solutions to (semilinear) elliptic and (quasilinear) hyperbolic equations.

RÉSUMÉ. – Dans cet article, nous considérons une suite de métriques lorentziennes $\{h_n\}_{n=1}^{+\infty}$, de classe C^4, satisfaisant les équations d'Einstein dans le vide $\mathrm{Ric}(h_n) = 0$. Nous supposons qu'il existe une métrique lorentzienne h_0 sur $\mathcal{M}$, de classe C^∞, telle que $h_n \to h_0$ uniformément sur tout compact. Nous supposons aussi que sur tout compact $K \subset \mathcal{M}$ il existe une suite de nombres strictement positifs $\lambda_n \to 0$ tels que

$$\|\partial^\alpha (h_n - h_0)\|_{L^\infty(K)} \lesssim \lambda_n^{1-|\alpha|}, \quad |\alpha| \geq 4.$$

Il est bien connu que h_0, qui représente une « limite haute-fréquence », n'est pas forcément solution des équations d'Einstein dans le vide. Cependant, il a été conjecturé par Burnett que h_0 devait être isométrique à une solution des équations d'Einstein couplées à un champ de Vlasov sans masse. Dans cet article, nous prouvons la conjecture de Burnett en supposant que $\{h_n\}_{n=1}^{+\infty}$ et h_0 admettent en plus une symétrie $\mathbb{U}(1)$ et satisfont une condition de jauge elliptique. La preuve utilise les mesures de défaut microlocales – on identifie une mesure de défaut microlocale définie de manière ad hoc comme étant la mesure de Vlasov dans l'espace-temps limite. Afin de montrer que cette mesure satisfait bien les équations de Vlasov, nous avons besoin d'annulations particulières qui reposent sur la structure

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précise des équations d'Einstein. Ces annulations sont liées à un nouveau phénomène de « compacité par compensation trilinéaire » pour des solutions d'un système couplant des équations elliptiques semi-linéaires à des équations hyperboliques quasilinéaires.

1. Introduction

It has been known in the context of classical general relativity that "backreaction of high frequency gravitational waves mimics effective matter fields" (see for instance [2, 3, 8, 9, 12, 13]). One way to describe this phenomenon mathematically (due to Burnett [2]) is to consider a sequence of (sufficiently regular) Lorentzian metrics $\{h_n\}_{n=1}^{+\infty}$ on a smooth manifold $\mathcal{M}$ satisfying the Einstein vacuum equations

$$\mathrm{Ric}(h_n) = 0 \tag{1.1}$$

such that (in some coordinate system) the metric components admit some limit h_0 where $h_n \to h_0$ uniformly on compact sets and $\partial h_n \to \partial h_0$ weakly. Assume moreover that for any compact set K, there is some sequence of positive numbers $\lambda_n \to 0$ such that the following holds on K:

$$|h_n - h_0| \lesssim \lambda_n, \quad |\partial h_n| \lesssim 1, \quad |\partial^k h_n| \lesssim \lambda_n^{-k+1} \text{ for } k = 2, 3, 4. \tag{1.2}$$

Due to the nonlinearity of the Einstein equations, the limit h_0 does not necessarily satisfy (1.1). Instead, in general it is possible for h_0 to satisfy

$$\mathrm{Ric}(h_0) - \frac{1}{2} h_0 R(h_0) = T$$

(where R is the scalar curvature) for some non-trivial stress-energy-momentum tensor T. This tensor T that arises in the limit can be interpreted as an effective matter field.

A question arises as to what type of effective matter field can arise in such a limiting process. In this direction, Burnett made the following conjecture [(1)]:

Conjecture 1.1 (Burnett [2]). – *Given $(\mathcal{M}, h_n)$ and $(\mathcal{M}, h_0)$ above, the limit h_0 is isometric to a solution to the Einstein-massless Vlasov system, i.e., the effective stress-energy-momentum tensor corresponds to that of massless Vlasov matter.*

We refer the reader to Sections 2.3–2.6 below for definitions concerning the Einstein-massless Vlasov system. We remark that in Conjecture 1.1, "Einstein-massless Vlasov system" has to be appropriately formulated to include *measure-valued* Vlasov fields since there are known examples for which the limits are isometric to solutions to the Einstein–null dust system. For further background on the Einstein–Vlasov system, see for instance [1, 15].

Conjecture 1.1 can be interpreted as stating that the effective matter field must be propagating with the speed of light and that the matter propagating in different directions do not directly interact, but only interact through their effect on the geometry; see [2].

Our main result is a proof of Conjecture 1.1 under two additional assumptions:

(1) We remark that in the original [2], (1.2) is only required to hold up to $k = 2$. We impose the slightly stronger assumption that (1.2) holds up to $k = 4$ in view of the result that we prove in this paper.

1. ($\mathbb{U}(1)$ symmetry.) The sequence $\{h_n\}_{n=1}^{+\infty}$ and the limit h_0 all admit a $\mathbb{U}(1)$ symmetry (without necessarily obeying a polarization condition).
2. (Elliptic gauge.) All the metrics can be put in an elliptic gauge and the bounds (1.2) hold in this gauge.

The following is our main theorem; see Theorem 4.2 for a precise statement.

THEOREM 1.2. – *Conjecture 1.1 is true under the above two additional assumptions.*

Theorem 1.2 implies a fortiori that the effective stress-energy-momentum tensor is traceless, obeys the *dominant* energy condition (i.e., for every future-directed causal vector X, the vector $-T^{\mu}{}_{\nu}X^{\nu}$ is also future-directed and causal), and is non-negative in the sense that $T(X,X) \geq 0$ pointwise for every vector field X (not necessarily causal). In fact, we show that these statements continue to hold even if we relax the convergence assumption to be significantly weaker than (1.2). We give an informal statement here but refer the reader to Theorem 4.1 for a precise statement.

THEOREM 1.3. – *Assume that h_n, h_0 all admit a* $\mathbb{U}(1)$ *symmetry and are put in an elliptic gauge. Suppose* (1.2) *is replaced by the conditions that $h_n \to h_0$ uniformly on compact sets and $\partial h_n \rightharpoonup \partial h_0$ weakly in $L^{p_0}_{\mathrm{loc}}$ for some $p_0 > \frac{8}{3}$.*

Then the effective stress-energy-momentum tensor is traceless, obeys the dominant energy condition, and is non-negative.

Theorem 1.3 can be compared with the following theorem of Green–Wald [8], which to our knowledge is so far the best result towards Conjecture 1.1:

THEOREM 1.4 (Green–Wald [8]). – *Assume $\{h_n\}_{n=1}^{+\infty}$ and h_0 are such that* (1.1) *and* (1.2) *hold. Then the effective stress-energy-momentum tensor is traceless and obeys the* weak *energy condition (i.e., $T(X,X) \geq 0$ pointwise for every timelike X).*

Note that while its conclusion is weaker than Theorem 1.3, Theorem 1.4 is a general result which does *not* require $\mathbb{U}(1)$ symmetry.

While our results are gauge-dependent, it should be mentioned that a large class of non-trivial examples have been constructed under our gauge conditions. In our previous paper [10], we have constructed sequences of solutions of Einstein vacuum equation with polarized $\mathbb{U}(1)$ symmetry, which can be put in an elliptic gauge, such that (1.2) are satisfied and the limit is a solution to Einstein equations coupled to N null dusts. See further discussions in Section 1.2.1.

We now briefly discuss the proof; for more details see Section 1.1. Under the $\mathbb{U}(1)$ symmetry assumptions, the $(3+1)$-dimensional Einstein vacuum equations reduce to the $(2+1)$-dimensional Einstein–wave map system. The rough strategy is the following:

— The first step of the proof is to show that only the two scalar fields corresponding to the wave map part of the system are responsible for the failure of the limit to be vacuum. This can already be viewed as a form of compensated compactness.