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PLATEAU PROBLEMS FOR MAXIMAL SURFACES IN PSEUDO-HYPERBOLIC SPACE

BY François LABOURIE, Jérémy TOULISSE AND Michael WOLF

ABSTRACT. – We define and prove the existence of unique solutions of an asymptotic Plateau problem for spacelike maximal surfaces in the pseudo-hyperbolic space of signature $(2,n)$: the boundary data is given by loops on the boundary at infinity of the pseudo-hyperbolic space which are limits of positive curves. We also discuss a compact Plateau problem. The required compactness arguments rely on an analysis of the pseudo-holomorphic curves defined by the Gauß lifts of the maximal surfaces.

RÉSUMÉ. – Nous définissons un problème de Plateau asymptotique pour les surfaces maximales de type espace dans l'espace pseudo-hyperbolique de signature $(2,n)$ dont le bord à l'infini est donné par des courbes, dites *semi—positives*, et qui sont limites de courbes positives. Nous montrons l'existence et l'unicité des solutions correspondantes et discutons le problème de Plateau compact correspondant. Les arguments de compacité utilisés requièrent l'analyse de courbes pseudo-holomorphes définies par le relevé de Gauß de surfaces maximales.

1. Introduction

Our goal is to study Plateau problems in the pseudo-hyperbolic space $\mathbf{H}^{2,n}$, which can be quickly described as the space of negative definite lines in a vector space of signature $(2,n+1)$. As such $\mathbf{H}^{2,n}$ is a complete homogeneous pseudo-Riemannian manifold of signature $(2,n)$ and curvature -1.

Quite naturally, $\mathbf{H}^{2,n}$ bears many resemblances to the hyperbolic plane, which corresponds to the case $n=0$. In particular, generalizing the Klein model, $\mathbf{H}^{2,n}$ may be described

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as one of the connected components of the complement to a quadric in the projective space of dimension $n+2$.

This quadric is classically called the *Einstein universe* and we shall denote it by $\partial_\infty \mathbf{H}^{2,n}$ [5]. Analogously to the hyperbolic case, the space $\partial_\infty \mathbf{H}^{2,n}$ carries a conformal metric of signature $(1,n)$ and we will consider it as a boundary at infinity of $\mathbf{H}^{2,n}$. Topologically, $\partial_\infty \mathbf{H}^{2,n}$ is the quotient of $\mathbf{S}^1 \times \mathbf{S}^n$ by an involution.

From the Lie group perspective, the space $\mathbf{H}^{2,n}$ has $\mathsf{PSO}(2,n+1)$ as a group of isometries and the Einstein space $\partial_\infty \mathbf{H}^{2,n}$ is the *Shilov boundary* of this rank two Hermitian group, that is the unique closed $\mathsf{PSO}(2,n+1)$-orbit in the boundary of the symmetric domain.

Positive triples and positivity in the Shilov boundary [19] play an important role in the theory of Hermitian symmetric spaces; of notable importance are the *positive loops*. Important examples of these are spacelike curves homotopic to $\mathbf{S}^1$ and specifically the *positive circles* which are boundaries at infinity in our compactification to totally geodesic embeddings of hyperbolic planes. Then *semi-positive loops* are limits of positive loops in some natural sense (see paragraph 2.5.2 for precise definitions).

Surfaces in a pseudo-Riemannian space may have induced metrics of variable signatures. We are interested in this paper in *spacelike surfaces* in which the induced metric is positive everywhere. Among these are the *maximal surfaces*, which are critical points of the area functional, for variations with compact support, see paragraph 3.3.3 for details. These maximal surfaces are the analogs of minimal surfaces in the Riemannian setting. An important case of those maximal surfaces in $\mathbf{H}^{2,n}$ is, again, the one of the totally geodesic surfaces which are isometric to hyperbolic planes.

We refer to the first two sections of this paper for precise definitions of what we have described only roughly above.

Our main Theorem is the following.

THEOREM A (Asymptotic Plateau problem). – *Any semi-positive loop in $\partial_\infty \mathbf{H}^{2,n}$ bounds a unique complete maximal surface in $\mathbf{H}^{2,n}$.*

In this paper, a semi-positive loop is not necessarily smooth. Also note that a properly embedded surface might not be complete and so the completeness condition is not vacuous.

On the other hand, we will show in section 3 that complete spacelike surfaces limit on semi-positive loops in $\partial_\infty \mathbf{H}^{2,n}$, and so Theorem A may be understood as identifying semi-positivity as the condition on curves in $\partial_\infty \mathbf{H}^{2,n}$ that corresponds to complete maximality for surfaces in $\mathbf{H}^{2,n}$.

The uniqueness part of the theorem is strikingly different from the corresponding setting in hyperbolic space where the uniqueness of solutions of the asymptotic Plateau problem fails in general for some quasi-symmetric curves as shown by Anderson, Wang and Huang [2, 44, 27].

As a tool in the theorem above, we also prove the following result, of independent interest, on the Plateau problem with boundary in $\mathbf{H}^{2,n}$. The relevant notion for curves is that of *strongly positive curves*, and among those the connected set of *deformable* ones which are defined in paragraph 3.2.

THEOREM B (Plateau problem). – *Any deformable strongly positive closed curve in $\mathbf{H}^{2,n}$ bounds a unique compact complete maximal surface in $\mathbf{H}^{2,n}$.*

One of the original motivations for this paper comes from the "equivariant situation." Recall that $\mathsf{G} := \mathsf{PSO}(2, n+1)$ is the isometry group of a Hermitian symmetric space M: the maximal compact subgroup of G has an $\mathsf{SO}(2)$ factor which is associated to a line bundle L over M. Thus a representation ρ of the fundamental group of a closed orientable surface S in G carries a *Toledo invariant*: the Chern class of the pull back of L by any map equivariant under ρ from the universal cover of S to M [43]. The *maximal representations* are those for which the integral of the Toledo invariant achieves its maximal value. These maximal representations have been extensively studied, from the point of view of Higgs bundles, by Bradlow, García-Prada and Gothen [13] and from the perspective of bounded cohomology, by Burger, Iozzi and Wienhard [15]. In particular, a representation is maximal if and only if it preserves a positive continuous curve [14, 15]. Then Collier, Tholozan and Toulisse have shown that there exists a unique equivariant maximal surface with respect to a maximal representation in $\mathsf{PSO}(2, n+1)$ [20]. This last result, an inspiration for our work, is now a consequence of Theorem A.

We note that maximal surfaces in $\mathbf{H}^{2,n}$ were also considered in a work by Ishihara [29]—see also Mealy [37]—and that Yang Li has obtained results for the finite Plateau problems in the Lorentzian case [35], while the codimension one Lorentzian case was studied by Bartnik and Simon in [6]. Yang Li's paper contains many references pertinent to the flat case. Neither paper restricts to two spacelike dimensions.

Another motivation comes from the contemplation of two other rank two groups: $\mathsf{SL}(3,\mathbb{R})$ and $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$, where we notice the latter group is isogenic to $\mathsf{PSO}(2,2)$.

Affine spheres and $\mathsf{SL}(3,\mathbb{R})$*:* While maximal surfaces are the natural conformal variational problem for $\mathsf{SO}(2,n)$, the analogous problem in the setting of $\mathsf{SL}(3,\mathbb{R})$ is that of affine spheres. Cheng and Yau [17], confirming a conjecture due to Calabi, proved that given any properly convex curve in the real projective space, there exists a unique affine sphere in $\mathbb{R}^3$ asymptotic to it. That result has consequences for the equivariant situation as well, due independently to Loftin and Labourie [36, 32]. Our main Theorem A may be regarded as an analogue of the Cheng—Yau Theorem: both affine spheres and maximal surfaces (for $\mathsf{SO}(2,3)$) are lifted as holomorphic curves—known as cyclic surfaces in [33]—in G/K_1, where G is $\mathsf{SL}(3,\mathbb{R})$ in the first case and $\mathsf{SO}(2,3)$ in the second, and K_1 is a compact torus. Moreover these holomorphic curves finally project as minimal surfaces in the symmetric space of G.

The case $n = 1$: The case of $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ and maximal surfaces in $\mathbf{H}^{2,1}$ has been extensively studied by Bonsante and Schlenker [10] and written only in the specific case of quasi-symmetric boundaries values—see also Tamburelli [40, 41] for further extensions. Our main Theorem A is thus a generalization in higher dimension of one of their main results. Also in the case of $\mathbf{H}^{2,1}$, we note that Bonsante and Seppi [11] have shown the existence, for any $K < -1$, of a unique K-surface extending a semi-positive loop in $\partial_\infty \mathbf{H}^{2,1}$.

Lorentzian asymptotic Plateau problem: There is also an interesting analogy with the work of Bonsante, Seppi and Smillie [12] in which they prove that, for every $H > 0$, any regular domain D in the $(n+1)$-dimensional Minkowski space contains a unique entire spacelike