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RELATIVE CRITICAL LOCI AND QUIVER MODULI

BY Tristan BOZEC, Damien CALAQUE
AND Sarah SCHEROTZKE

ABSTRACT. – In this paper we identify the cotangent stack to the derived stack of representations of a quiver Q with the derived moduli stack of modules over the Ginzburg dg-algebra associated with Q. More generally, we extend this result to finite type dg-categories, to a relative setting as well, and to deformations of these. It allows us to recover and generalize some results of Yeung, and leads us to the discovery of seemingly new Lagrangian subvarieties in the Hilbert scheme of points in the plane.

RÉSUMÉ. – Dans cet article nous identifions le champ cotangent du champ dérivé des représentations d'un carquois Q avec le champ dérivé des modules sur la dg-algèbre de Ginzburg associée à Q. Nous étendons ce résultat aux dg-catégories de type fini, ainsi qu'à un cadre relatif, et à des déformations de ce qui précède. Cela nous permet de retrouver et généraliser des résultats de Yeung, et nous mène à la découverte de nouvelles sous-variétés lagrangiennes du schéma de Hilbert de points dans le plan.

1. Introduction

This paper is concerned with Calabi-Yau structures on smooth dg-categories and functors thereof, as well as with symplectic and Lagrangian structures on derived Artin stacks and morphisms between these. Its main goal can be summarized as giving a precise meaning (and a proof) to the claim that deformations of Calabi-Yau completions of dg-categories (resp. relative Calabi-Yau completions) are a non-commutative analog of deformations of cotangent stacks (resp. conormal stacks).

A guiding idea of noncommutative (algebraic) geometry is that a noncommutative analog of a standard geometric notion T is some algebraic notion T^{nc} that induces functorially a structure of type T on schemes of representations (see e.g., [14] and references therein). It has been advocated by several authors that the noncommutative analog of the cotangent bundle should be obtained as a free construction, and that, in the case of quivers, if one applies a noncommutative version of hamiltonian reduction, then one recovers the preprojective algebra (see e.g., [10]). We would like to amend this approach in several ways:

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- First of all, we would like to replace representation schemes by moduli stacks of representations, so that the preprojective algebra itself becomes the correct noncommutative cotangent.
- Then, in order for most functors to remain well-behaved, we work within a homotopy invariant framework, and so we will in fact consider the *derived* moduli of representations. The preprojective algebra will then be replaced by its differential graded variant, introduced by Ginzburg [13].
- Finally, we will consider the whole derived moduli of perfect dg-modules, also called *moduli of objects*, after Toën-Vaquié [40]. This also has the advantage that it extends to dg-categories.

In 2011, Keller [23] introduced a broad generalization of Ginzburg dg-algebras: the so-called Calabi-Yau completions of smooth dg-categories (which, surprisingly enough, are indeed given as a free construction), and their deformations. Recently, Yeung [43] advocated that Calabi-Yau completions should be viewed as noncommutative cotangent bundles. It is actually known, after Toën [38, 39] and Brav-Dyckerhoff [4], that the moduli of objects functor **Perf** sends finite type n-Calabi-Yau dg-categories to $(2-n)$-shifted symplectic stacks in the sense of Pantev-Toën-Vaquié-Vezzosi [31]. In other words, Calabi-Yau structures on dg-categories are the non-commutative (shifted) symplectic structures. In this paper, we prove the following.

Theorem (Theorem 6.17). – *For a finite type dg-category $\mathscr{A}$, there is an equivalence of exact $(2-n)$-shifted symplectic stacks between the shifted cotangent stack $\mathbf{T}^*[2-n]\mathbf{Perf}_{\mathscr{A}}$ and the perfect moduli $\mathbf{Perf}_{\mathcal{G}_n(\mathscr{A})}$ of the n-Calabi-Yau completion $\mathcal{G}_n(\mathscr{A})$ of $\mathscr{A}$.*

We also prove a deformed version of the above, identifying the moduli of objects of deformed Calabi-Yau completions with twisted cotangent stacks, as shifted symplectic stacks.

Our major geometric motivation lies in the construction of Lagrangian (rather than just symplectic) structures, and their connections with critical loci or Calabi-Yau geometry. To be more specific, inspired by classical symplectic geometry [33], we are interested in deformations of conormal stacks that appear as *relative critical loci* (see [7, 1]). In certain cases, these relative critical loci can be obtained as moduli of representations of relative versions of Ginzburg dg-algebras. If we use the formalism of derived symplectic geometry, introduced by Pantev-Toën-Vaquié-Vezzosi [31], we will show that our constructions yield very down-to-earth examples, such as Lagrangian subvarieties of some quiver moduli spaces. Moduli of quiver representations form indeed a large and fruitful class of applications in this area, and will serve as an ongoing example in the present study. More generally, moduli of objects of finite type dg-categories, as defined by Toën-Vaquié [40], encompass quiver representations and provide a lot more examples to play with. In fact, as we have already seen, it is essential in the statement itself of the results that achieve our main goal.

Elaborating on an idea of Toën [38], Brav-Dyckerhoff [3] have introduced a very useful notion of relative (left) Calabi-Yau structure on a dg-functor, and have shown [4] (see also [39]) that these relative Calabi-Yau structures are the correct noncommutative analog of Lagrangian structures on morphisms between derived stacks: the moduli of objects functor

Perf sends functors (between finite type dg-categories) equipped with a relative Calabi-Yau structure to morphisms (between derived Artin stacks) equipped with a Lagrangian structure. Soon after, Yeung [44] introduced a relative analog of Keller's Calabi-Yau completions along functors between finite cellular dg-categories, and showed that they indeed carry a relative Calabi-Yau structure in the sense Brav-Dyckerhoff.

In this paper, with every dg-functor $f : \mathscr{A} \to \mathscr{B}$ between smooth dg-categories we associate a cospan

$$\mathcal{G}_n(\mathscr{A}) \longrightarrow \mathcal{G}_n(f) \longleftarrow \mathcal{G}_n(\mathscr{B})$$

and we prove the following.

THEOREM (Theorem 5.23 & Theorem 6.18). – (1) *The above cospan carries an exact n-Calabi-Yau structure.*

(2) *For finite type dg-categories, the moduli of objects functor sends this exact n-Calabi-Yau cospan to the exact Lagrangian correspondence*

$$\mathbf{T}^*[2-n]\mathbf{Perf}_{\mathscr{A}} \longleftarrow \varphi^*\mathbf{T}^*[2-n]\mathbf{Perf}_{\mathscr{A}} \longrightarrow \mathbf{T}^*[2-n]\mathbf{Perf}_{\mathscr{B}},$$

where $\varphi = f^* : \mathbf{Perf}_{\mathscr{B}} \to \mathbf{Perf}_{\mathscr{A}}$.

As a consequence of the first point, we recover Yeung's deformed relative Calabi-Yau completion (without the finite cellular assumption) thanks to a simple use of the composition of Calabi-Yau cospans. As a consequence of the second point, we obtain that the moduli of objects of certain deformed relative Calabi-Yau completions can be identified with relative critical loci. We understand very concrete examples of these relative critical loci in the case of quiver representations and exhibit in particular some new (to our knowledge) Lagrangian subvarieties of the Hilbert schemes of points on the plane.

We now summarize the new contributions of this paper:

- We extend several results of Yeung on the existence of (relative) Calabi-Yau structures from finite cellular dg-categories to smooth dg-categories, simplifying the proofs.
- The simplification is made possible thanks to a new exact Calabi-Yau cospan associated with every dg-functor $f : \mathscr{A} \to \mathscr{B}$, that we also use to identify the Calabi-Yau completion with a noncommutative hamiltonian reduction.
- We identify the moduli stack of objects of a (relative) deformed Calabi-Yau completion with the (relative) critical locus.
- In the case of the inclusion of the one loop quiver in its tripled version, we discover new Lagrangian subvarieties in the Hilbert scheme of $\mathbb{C}^2$.

Outline of the paper