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MOTIVIC CHERN CLASSES OF SCHUBERT CELLS, HECKE ALGEBRAS, AND APPLICATIONS TO CASSELMAN'S PROBLEM

BY Paolo ALUFFI, Leonardo C. MIHALCEA, Jörg SCHÜRMANN AND Changjian SU

ABSTRACT. – Motivic Chern classes are elements in the K-theory of an algebraic variety X, depending on an extra parameter y. They are determined by functoriality and a normalization property for smooth X. In this paper we calculate the motivic Chern classes of Schubert cells in the (equivariant) K-theory of flag manifolds G/B. We show that the motivic class of a Schubert cell is determined recursively by the Demazure-Lusztig operators in the Hecke algebra of the Weyl group of G, starting from the class of a point. The resulting classes are conjectured to satisfy a positivity property. We use the recursions to give a new proof that they are equivalent to certain K-theoretic stable envelopes recently defined by Okounkov and collaborators, thus recovering results of Fehér, Rimányi and Weber. The Hecke algebra action on the K-theory of the Langlands dual flag manifold matches the Hecke action on the Iwahori invariants of the principal series representation associated to an unramified character for a group over a nonarchimedean local field. This gives a correspondence identifying the duals of the motivic Chern classes to the standard basis in the Iwahori invariants, and the fixed point basis to Casselman's basis. We apply this correspondence to prove two conjectures of Bump, Nakasuji and Naruse concerning factorizations and holomorphy properties of the coefficients in the transition matrix between the standard and the Casselman's basis.

RÉSUMÉ. – Les *classes de Chern motiviques* sont des éléments de la K-théorie d'une variété algébrique X, qui dépendent d'un paramètre supplémentaire y. Elles sont déterminées par la fonctorialité et une propriété de normalisation pour X lisse. Dans cet article, nous calculons les classes de Chern motiviques des cellules de Schubert dans la K-théorie (équivariante) des variétés de drapeaux G/B. Nous montrons que la classe motivique d'une cellule de Schubert est détérminée récursivement grâce aux opérateurs de Demazure-Lusztig de l'algèbre de Hecke du groupe de Weyl de G, à partir de la classe d'un point. Nous conjecturons que les classes obtenues satisfont une propriété de positivité. Nous utilisons nos récurrences pour obtenir une nouvelle preuve du fait que les classes sont équivalentes à certaines enveloppes stables définies récemment en K-théorie par Okounkov et ses collaborateurs, retrouvant ainsi un résultat de Fehér, Rimányi, et Weber. L'action de l'algèbre de Hecke sur la K-théorie

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de la variété de drapeaux du dual de Langlands coïncide avec l'action de Hecke sur les invariants d'Iwahori de la représentation par série principale associée à un caractère non ramifié pour un groupe sur un corps local non archimédien. Cela induit une correspondance identifiant les duaux des classes de Chern motiviques à la base standard du module des invariants d'Iwahori, et la base des points fixes à la base de Casselman. Nous appliquons ce résultat pour démontrer deux conjectures dues à Bump, Nakasuji et Naruse concernant les factorisations et les propriétés d'holomorphie des coefficients de la matrice de transition entre la base standard et la base de Casselman.

1. Introduction

Let X be a complex algebraic variety, and let $K_0(\mathrm{var}/X)$ be the (relative) Grothendieck group of varieties over X. It consists of classes of morphisms $[f : Z \to X]$ modulo the scissors relations; cf. [58, 14] and §4 below. Brasselet, Schürmann and Yokura [17] defined the *motivic Chern transformation* $\mathrm{MC}_y : K_0(\mathrm{var}/X) \to K(X)[y]$ with values in the K-theory group of coherent sheaves in X to which one adjoins a formal variable y. The transformation MC_y is a group homomorphism, it is functorial with respect to proper push-forwards, and if X is smooth, it satisfies the normalization condition

$$\mathrm{MC}_y[\mathrm{id}_X : X \to X] = \sum [\wedge^j T^*(X)] y^j.$$

Here $[\wedge^j T^*(X)]$ is the K-theory class of the bundle of degree j differential forms on X. If $Z \subseteq X$ is a constructible subset, we denote by $\mathrm{MC}_y(Z) := \mathrm{MC}_y[Z \hookrightarrow X] \in K(X)[y]$ the motivic Chern class of Z. Because MC_y is a group homomorphism, it follows that if $X = \bigsqcup Z_i$ is a disjoint union of constructible subsets, then $\mathrm{MC}_y(X) = \sum \mathrm{MC}_y(Z_i)$. As explained in [17], the motivic Chern class $\mathrm{MC}_y(Z)$ is related by a Hirzebruch-Riemann-Roch type statement to the Chern-Schwartz-MacPherson (CSM) class $c_{\mathrm{SM}}(Z)$ in the homology of X. We recall that the existence and functoriality properties of this CSM class were conjectured by Deligne and Grothendieck and proved by Robert MacPherson ([61]). Earlier, Marie-Hélène Schwartz had independently established a theory of Chern classes for singular varieties, using obstruction theory ([79, 80]). Jean-Paul Brasselet and Schwartz proved that the two classes coincide via the Alexander isomorphism ([18]). Both the motivic and the CSM classes give a functorial way to attach K-theory, respectively (co)homology classes, to constructible subsets, and both satisfy the usual motivic relations. There is also an equivariant version of the motivic Chern class transformation, which uses equivariant varieties and morphisms, and has values in the appropriate equivariant K-theory group. Its definition was given in [38], following closely the approach of [17].

We take this opportunity to provide further details on the construction and properties of equivariant motivic Chern classes, such as functoriality and a Verdier-Riemann-Roch formula; see §4 below. However, the main goals of this paper are to build the computational foundations for the study of the (equivariant) motivic Chern classes of Schubert cells in the generalized flag manifolds, and to relate this to the representation theory of p-adic groups. Our main application consists of formulas for the transition coefficients between the standard and the Casselman bases of the module of Iwahori invariants of the principal series representation, in terms of localization of motivic Chern classes.

Let G be a complex, semisimple, linear algebraic group, and B a Borel subgroup. By functoriality, the (equivariant) motivic Chern classes of Schubert cells in G/B determine those in any flag manifold G/P, where P is a parabolic subgroup. Based on previously discovered features of the CSM classes of Schubert cells [8, 75, 5], it was expected that the motivic classes would be closely related to objects which appear in geometric representation theory. We prove in this paper that the motivic Chern classes of Schubert cells are recursively determined by the Demazure-Lusztig operators which appear in early works of Lusztig on Hecke algebras [60]. Further, the motivic classes of Schubert cells are equivalent (in a precise sense) to the K-theoretic stable envelopes defined by Okounkov and collaborators in [71, 3, 70]. This equivalence was proved recently by Fehér, Rimányi and Weber [38]; our approach, based on comparing the Demazure-Lusztig recursions to the recursions for the stable envelopes found by Su, Zhao and Zhong in [84] gives another proof of this result. Via this equivalence, the motivic Chern classes can be considered as natural analogues of the Schubert classes in the K-theory of the cotangent bundle of G/B.

As in the authors' previous work on CSM classes [5], the connections to Hecke algebras and K-theoretic stable envelopes yield remarkable identities among (duals of) motivic Chern classes. We use these identities to prove two conjectures of Bump, Nakasuji and Naruse [26, 27, 66] about the coefficients in the transition matrix between the Casselman's basis and the standard basis in the Iwahori-invariant space of the principal series representation for an unramified character for a group over a non archimedean local field.

We present next a more extensive description of our results.

1.1. Statement of results

Let G be a complex, semisimple, linear algebraic group, and fix B, B^- a pair of opposite Borel subgroups of G. Denote by $T := B \cap B^-$ the maximal torus, by $W := N_G(T)/T$ the Weyl group, and by $X := G/B$ the (generalized) flag variety. For each Weyl group element $w \in W$ consider the Schubert cell $X(w)^\circ := BwB/B$, a subvariety of (complex) dimension $\ell(w)$. The opposite Schubert cell $Y(w)^\circ := B^-wB/B$ has complex codimension $\ell(w)$. The closures $X(w)$ and $Y(w)$ of these cells are the Schubert varieties. Let $\mathcal{O}_w$, respectively $\mathcal{O}^w$ be the K-theoretic Schubert classes associated to the structure sheaves of $X(w)$, respectively $Y(w)$. The equivariant K-theory ring of X, denoted by $K_T(X)$, is an algebra over $K_T(\mathrm{pt}) = R(T)$—the representation ring of T—and it has an $R(T)$-basis given by the Schubert classes $\mathcal{O}_w$ (or $\mathcal{O}^w$), where w varies in the Weyl group W.

If E is an equivariant vector bundle over X, we denote by $[E]$ its class in $K_T(X)$, and by $\lambda_y(E)$ the class

$$\lambda_y(E) = \sum [\wedge^i E] y^i \in K_T(X)[y].$$

For a T-stable subvariety $\Omega \subseteq X$ recall the notation

$$\mathrm{MC}_y(\Omega) := \mathrm{MC}_y[\Omega \hookrightarrow X] \in K_T(X)[y].$$

Our first main result is a recursive formula to calculate $\mathrm{MC}_y(X(w)^\circ)$, the (equivariant) motivic Chern class of the Schubert cell. For each simple positive root α_i, consider the