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ARITHMETIC GEOMETRY OF TORIC VARIETIES. METRICS, MEASURES AND HEIGHTS

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ARITHMETIC GEOMETRY OF TORIC VARIETIES. METRICS, MEASURES AND HEIGHTS

José Ignacio Burgos Gil, Patrice Philippon, Martín Sombra

Abstract. — We show that the height of a toric variety with respect to a toric metrized line bundle can be expressed as the integral over a polytope of a certain adelic family of concave functions. To state and prove this result, we study the Arakelov geometry of toric varieties. In particular, we consider models over a discrete valuation ring, metrized line bundles, and their associated measures and heights. We show that these notions can be translated in terms of convex analysis, and are closely related to objects like polyhedral complexes, concave functions, real Monge-Ampère measures, and Legendre-Fenchel duality.

We also present a closed formula for the integral over a polytope of a function of one variable composed with a linear form. This formula allows us to compute the height of toric varieties with respect to some interesting metrics arising from polytopes. We also compute the height of toric projective curves with respect to the Fubini-Study metric and the height of some toric bundles.

Résumé **(Géométrie arithmétique des variétés toriques. Métriques, mesures et hauteurs)**

Nous montrons que la hauteur d'une variété torique relative à un fibré en droites métrisé torique s'écrit comme l'intégrale sur un polytope d'une certaine famille adélique de fonctions concaves. Afin d'énoncer et démontrer ce résultat, nous étudions la géométrie d'Arakelov des variétés toriques. En particulier, nous considérons des modèles de ces variétés sur des anneaux de valuation discrète, ainsi que les fibrés en droites métrisés et leurs mesures et hauteurs associées. Nous montrons que ces notions se traduisent en termes d'analyse convexe et sont intimement liées à des objets tels que les complexes polyhédraux, les mesures de Monge-Ampère et la dualité de Legendre-Fenchel.

Nous présentons également une formule close pour l'intégration sur un polytope d'une fonction d'une variable composée avec une forme linéaire. Cette formule nous permet de calculer la hauteur de variétés toriques relativement à plusieurs métriques intéressantes, provenant de polytopes. Nous calculons aussi la hauteur des courbes toriques projectives relativement à la métrique de Fubini-Study et la hauteur des fibrés toriques.

CONTENTS

INTRODUCTION

Systems of polynomial equations appear in a wide variety of contexts in both pure and applied mathematics. Systems arising from applications are not random but come with a certain structure. When studying those systems, it is important to be able to exploit that structure.

A relevant result in this direction is the Bernštein-Kušnirenko-Khovanskii theorem [**Kuš76, Ber75**]. Let K be a field with algebraic closure $\overline{K}$. Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope and $f_1, \dots, f_n \in K[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ a family of Laurent polynomials whose Newton polytope is contained in Δ. The BKK theorem says that the number (counting multiplicities) of isolated common zeros of $f_1, \dots, f_n$ in $(\overline{K}^{\times})^n$ is bounded above by $n!$ times the volume of Δ, with equality when $f_1, \dots, f_n$ is generic among the families of Laurent polynomials with Newton polytope contained in Δ. This shows how a geometric problem (the counting of the number of solutions of a system of equations) can be translated into a combinatorial, simpler one. It is commonly used to predict when a given system of polynomial equations has a small number of solutions. As such, it is a cornerstone of polynomial equation solving and has motivated a large amount of work and results over the past 25 years, see for instance [**GKZ94, Stu02, PS08b**] and the references therein.

A natural way to study polynomials with prescribed Newton polytope is to associate to the polytope Δ a toric variety X over K equipped with an ample line bundle L. The polytope conveys all the information about the pair (X, L). For instance, the degree of X with respect to L is given by the formula

$$\deg_L(X) = n! \operatorname{vol}(\Delta),$$

where vol denotes the Lebesgue measure of $\mathbb{R}^n$. The Laurent polynomials f_i can be identified with global sections of L, and the BKK theorem can be deduced from this formula. Indeed, there is a dictionary which allows to translate algebro-geometric properties of toric varieties in terms of combinatorial properties of polytopes and fans, and the degree formula above is one entry in this “toric dictionary”.

The central motivation for this text is an arithmetic analogue for heights of this formula, which is the theorem stated below. The height is a basic arithmetic invariant of a proper variety over the field of rational numbers. Together with its degree, it measures the amount of information needed to represent this variety, for instance, *via* its Chow form. Hence, this invariant is also relevant in computational algebraic geometry, see for instance [**GHH**$^{+}$**97, AKS07, DKS12**]. The notion of height of varieties generalizes the height of points already considered by Siegel, Northcott, Weil and others, it is an essential tool in Diophantine approximation and geometry.

For simplicity of the exposition, in this introduction we assume that the pair (X, L) is defined over the field of rational numbers $\mathbb{Q}$, although in the rest of the book we will work with more general adelic fields (Definition 1.5.1). Let $\mathfrak{M}_{\mathbb{Q}}$ denote the set of places of $\mathbb{Q}$ and let $(\vartheta_v)_{v\in\mathfrak{M}_{\mathbb{Q}}}$ be a family of concave functions on Δ such that $\vartheta_v \equiv 0$ for all but a finite number of v. We will show that, to this data, one can associate an adelic family of metrics $(\|\cdot\|_v)_v$ on L. Write $\overline{L} = (L, (\|\cdot\|_v)_v)$ for the resulting metrized line bundle.

Theorem. — *The height of X with respect to $\overline{L}$ is given by*

$$\mathrm{h}_{\overline{L}}(X) = (n+1)! \sum_{v\in\mathfrak{M}_{\mathbb{Q}}} \int_{\Delta} \vartheta_v \,\mathrm{d}\,\mathrm{vol}\,.$$

This theorem was announced in [**BPS09**] and we prove it in the present text. To establish it in a wide generality, we have been led to study the Arakelov geometry of toric varieties. In the course of our research, we have found that a large part of the arithmetic geometry of toric varieties can be translated in terms of convex analysis. In particular, we have added a number of new entries to the arithmetic geometry chapter of the toric dictionary, including models of toric varieties over a discrete valuation ring, metrized line bundles, and their associated measures and heights. These objects are closely related to objects of convex analysis like polyhedral complexes, concave functions, Monge-Ampère measures and Legendre-Fenchel duality.

These additions to the toric dictionary are very concrete and well-suited for computations. In particular, they provide a new wealth of examples in Arakelov geometry where constructions can be made explicit and properties tested. In relation with explicit computations in these examples, we present a closed formula for the integral over a polytope of a function of one variable composed with a linear form. This formula allows us to compute the height of toric varieties with respect to some interesting metrics arising from polytopes. Some of these heights are related to the average entropy of a simple random process on the polytope. We also compute the height of toric projective curves with respect to the Fubini-Study metric and of some toric bundles.

There are many other arithmetic invariants of toric varieties that may be studied in terms of convex analysis. For instance, in the subsequent paper [**BMPS12**] we give