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MASSLESS PHASES FOR THE VILLAIN MODEL IN $d \geq 3$

by Paul DARIO & Wei WU

Abstract. — A major open question in statistical mechanics, known as the Gaussian spin wave conjecture, predicts that the low temperature phase of the Abelian spin systems with continuous symmetry behave like Gaussian free fields. In this paper we consider the classical Villain rotator model in $\mathbb{Z}^d$, $d \geq 3$ at sufficiently low temperature, and prove that the truncated two-point function decays asymptotically as $|x|^{2-d}$, with an algebraic rate of convergence. We also obtain the same asymptotic decay separately for the transversal two-point functions. This quantifies the spontaneous magnetization result for the Villain model at low temperatures and constitutes a first step toward a more precise understanding of the spin-wave conjecture. We believe that our method extends to finite range interactions, and to other Abelian spin systems and Abelian gauge theory in $d \geq 3$. We also develop a quantitative perspective on homogenization of uniformly convex gradient Gibbs measures.

Résumé. **(Phases sans masse du modèle de Villain pour $d \geq 3$)** – Une question ouverte majeure en mécanique statistique, connue sous le nom de conjecture des vagues de spins, prédit que les systèmes de spins équipés d'une symétrie abélienne continue se comportent comme des champs libres gaussiens à basse température. Dans cet article, nous considérons le modèle de Villain en dimension supérieure ou égale à 3 à une température suffisamment basse, et nous démontrons que la fonction de deux points décroît asymptotiquement comme celle d'un champ libre gaussien. Afin d'obtenir ce résultat, nous développons une approche quantitative pour l'homogénéisation des mesures de Gibbs sur les champs de gradients avec un potentiel uniformément convexe.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Rotator models and the spin wave picture

Rotator models, such as the XY and the Villain models, have drawn considerable attention from distinct research communities in mathematical and theoretical physics. They are of much interest in statistical mechanics, as they exhibit new types of phase transition for ferromagnetic systems and can be applied to the design of novel materials. A canonical rotator model is the XY model defined as follows: given a finite set $U \subseteq \mathbb{Z}^d$, we assign to each function $\theta : U \to (-\pi, \pi]$ satisfying $\theta = 0$ on the external vertex boundary ∂U the energy

$$H_U^{XY}(\theta) := -\sum_{\substack{x,y\in U^+ \\ x\sim y}} \cos(\theta(x) - \theta(y)),$$

where $U^+ := U \cap \partial U$ and the notation $x \sim y$ means that the points x and y are nearest neighbor in the lattice $\mathbb{Z}^d$. The Gibbs measure of the XY model with zero boundary condition at inverse temperature $\beta > 0$ is then defined the probability distribution

$$d\mu_{\beta,U}^{XY}(d\theta) := \frac{1}{Z_{\beta,U}^{XY}} \exp\left(-\beta H_U^{XY}(\theta)\right) \prod_{x\in U} \mathrm{d}\theta(x) 1_{\theta|_{\partial U}=0}. \tag{1.1.1}$$

The XY model can be equivalently seen as a spin system with spin valued in the circle $\mathbb{S}^1$ by setting $S_x := e^{i\theta_x}$. In this article, we will be interested in another closely related rotator model, the Villain model [**90**] is defined by the Gibbs weight

$$d\mu_{\beta,U}^{\mathrm{Vill}}(d\theta) := \frac{1}{Z_U^{\mathrm{Vill}}} \prod_{x\sim y} v_\beta(\theta(x) - \theta(y))\, \mathrm{d}\theta(x) 1_{\theta|_{\partial U}=0}, \tag{1.1.2}$$

where is the heat kernel on $\mathbb{S}^1$ defined according to the identity

$$v_\beta(\theta) := \sum_{m\in\mathbb{Z}} \exp\left(-\frac{\beta}{2}(\theta + 2\pi m)^2\right). \tag{1.1.3}$$

The two models belong to the class of spin systems with continuous Abelian symmetry. They exhibit a similar behavior and have been extensively studied in the literature. We collect below some of their main features.

Since the spins take values in the compact space $\mathbb{S}^1$, the existence of a thermodynamic limit for the XY model (i.e., an infinite-volume limit as $U \to \infty$) is guaranteed along subsequences by standard compactness arguments. It is additionally known that this limit is unique, and we denote it by μ_β^{XY} (see **[79]**). The Griffiths correlation inequalities **[58, 25, 79]** imply that that the expected value of the spins and the two-point function are monotone in the domain U and in particular show the convergences

$$\langle S_x \rangle_{\mu_{\beta,U}^{XY}} \underset{U \uparrow \mathbb{Z}^d}{\longrightarrow} \langle S_x \rangle_{\mu_\beta^{XY}} \qquad \text{and} \qquad \langle S_x \cdot S_y \rangle_{\mu_{\beta,U}^{XY}} \underset{U \uparrow \mathbb{Z}^d}{\longrightarrow} \langle S_x \cdot S_y \rangle_{\mu_\beta^{XY}} .$$

The same results hold for the Villain model, and we denote by μ_β^V the corresponding thermodynamic limit (1).

In two dimensions, the Mermin-Wagner theorem **[78]** shows that there is no continuous symmetry breaking at any temperature, i.e., for any $\beta > 0$,

$$\langle S_x \rangle_{\mu_\beta^{XY}} = 0. \tag{1.1.4}$$

In particular, the system does not undergo an order/disorder phase transition. Nevertheless, the system is known to exhibit a phase transition of a different type, characterized by a different asymptotic behavior of the correlation function: there exists a critical inverse temperature $\beta_c \in (0, \infty)$ such that in the low temperature regime $(\beta > \beta_c)$, the two-point function $\langle S_x \cdot S_0 \rangle_{\mu_\beta}$ decays polynomially fast (which characterizes a so-called topological long-range order **[72]**), while, in the high temperature regime $(\beta \le \beta_c)$, the two-point function decays exponentially fast. This phase transition is known as the Berezinskii-Kosterlitz-Thouless transition became the basis of the Nobel prize in Physics in 2016 to Haldane, Kosterlitz and Thouless. From a mathematical perspective, the existence of this transition was established in the celebrated work of Fröhlich and Spencer **[50]**, and has been the subject of recent developments **[75, 43, 4]**.

In the low temperature regime $(\beta > \beta_c)$, additional predictions can be made regarding the behavior of the model. A simple heuristics suggests that, as the temperature goes to zero, the spins tend to align with each other so as to minimize the Hamiltonian. Using the approximations, when $|\delta\theta| \ll 1$,

$$\exp(\beta \cos(\delta\theta)) \approx \sum_{m \in \mathbb{Z}} \exp\left(-\frac{\beta}{2}(\delta\theta + 2\pi m)^2\right) \qquad \text{and} \qquad \cos\left(2\pi\left(\delta\theta\right)\right) \approx 1 - \left(\delta\theta\right)^2/2, \tag{1.1.5}$$

it is expected that at low temperature, both the XY and the Villain Gibbs measures on large scales behave like the Gaussian measure

$$\mu_\beta^{GFF}(d\phi) := \frac{1}{Z} \exp\left(-\frac{\beta}{2} \sum_{x \sim y} (\phi(x) - \phi(y))^2\right) \prod_x d\phi(x). \tag{1.1.6}$$

1. The monotonicity of the correlation function and the uniqueness of the infinite volume Gibbs state were first established for the XY model **[79]**. However, the Villain model can be represented as a metric graph limit of the XY model **[85, 50]**. By taking this limit, we obtain the corresponding monotonicity and the uniqueness of Gibbs state for the Villain model.

The Gibbs measure (1.1.6) is the Gaussian free field, and its law is fully characterized by its covariance matrix given by the lattice Green's function. This heuristic computation is the starting point of the celebrated spin wave picture originating in the work of Dyson **[42]** (see also **[78]**). The spin wave conjecture predicts that at low temperatures both the XY and the Villain Gibbs measures behave on large scales like a Gaussian free field of the form (1.1.6) with a notable difference: since the approximations (1.1.5) are not exact (and does not recover the information of the periodized field in (1.1.1) and (1.1.2)), a corrective term, corresponding to the so-called vortex lines, has to be taken into account in the analysis, and the limiting Gaussian free field describing the large-scale behaviors of the XY and Villain models should display an effective temperature $\beta_{\mathrm{eff}} \neq \beta$ (with $\beta_{\mathrm{eff}} = (1+o(1))\beta$ as $\beta \to \infty$).

More precisely, the spin wave picture in the case of the two-point function asserts that, for $d=2$ and $\beta > \beta_c$, there exists an effective inverse temperature $\beta_{\mathrm{eff}} > 0$ (with $\beta_{\mathrm{eff}} \neq \beta$) such that

(1.1.7)
$$\left\langle e^{i(\theta(0)-\theta(x))} \right\rangle_{\mu_\beta^V} = \left\langle e^{i(\phi(0)-\phi(x))} \right\rangle_{\mu_{\beta_{\mathrm{eff}}}^{GFF}} (1+o(1)) = |x|^{-\frac{1}{2\pi\beta_{\mathrm{eff}}}} + o\left(|x|^{-\frac{1}{2\pi\beta_{\mathrm{eff}}}}\right).$$

Rigorous (but non-optimal) power law upper and lower bounds for the two-point function were established in the 1980s in the celebrated works of McBryan-Spencer **[77]** and Fröhlich-Spencer **[49]** in the low temperature regime, namely, for $\beta \gg 1$,

$$c_1 |x|^{-\frac{1}{2\pi\beta_1}} \leq \langle S_0 \cdot S_x \rangle_{\mu_\beta^V} \leq c_2 |x|^{-\frac{1}{2\pi\beta}},$$

where $\beta_1 = \beta_1(\beta)$ and satisfies $\beta_1 = \beta(1+o(1))$ as $\beta \to \infty$. For a closely related model, the two dimensional two-component Coulomb gas with small activity, Falco justified the spin wave picture (with an effective β_{eff} in the exponent) for all the inverse temperatures in the Kosterlitz-Thouless phase, in a series of impressive works **[45, 46]**. For the two dimensional XY and Villain models, the asymptotic two point function (1.1.7) still remains an important open question.

In three dimensions and higher, the breakthrough work of Fröhlich, Simon and Spencer **[48]** shows that these models undergo an order/disorder phase transitions: there exists an critical inverse temperature $\beta_c > 0$ such that

$$\text{for any } \beta > \beta_c, \quad \langle S_x \rangle_{\mu_\beta} \neq 0 \quad \text{and} \quad \text{for any } \beta < \beta_c, \quad \langle S_x \rangle_{\mu_\beta} = 0.$$

In the low temperature phase ($\beta > \beta_c$), the spin wave picture predicts that there exist two coefficients c_1, c_2 such that

$$\langle S_0 \cdot S_x \rangle_{\mu_\beta^V} = c_1 + \frac{c_2}{|x|^{d-2}} + o\left(\frac{1}{|x|^{d-2}}\right). \tag{1.1.8}$$

Considerable progress towards quantitative information for the XY/Villain model at low temperature were made in the 1980s. In dimensions $d \geq 3$, the best known result is the one of Fröhlich and Spencer **[51]** who observed that the classical Villain model in $\mathbb{Z}^d$ can be mapped, via duality, to a statistical mechanical model of lattice

Coulomb gas. They obtained the following next order description of the correlation function at low temperature.

***Proposition 1.1.1* (Fröhlich and Spencer [51]). —** *Let μ_β^V be the thermodynamic limit of the Villain model in $\mathbb{Z}^d$, for $d \geq 3$. There exist constants $\beta_0 = \beta_0(d)$, $c_0 = c_0(\beta, d)$, such that for all $\beta > \beta_0$,*

$$\langle S_0 \cdot S_x \rangle_{\mu_\beta^V} = c_0 + O\left(\frac{1}{|x|^{d-2}}\right).$$

Moreover, denote by G the lattice Green's function in $\mathbb{Z}^d$, then we have as $\beta \to \infty$,

$$\exp\left(\frac{1}{\beta}(G(0) - G(x))\right) \geq \langle S_0 \cdot S_x \rangle_{\mu_\beta^V} \geq \exp\left(\left(\frac{1}{\beta} + o\left(\frac{1}{\beta}\right)\right)(G(0) - G(x))\right).$$

This suggests that the truncated two-point function may be related to a massless free field in $\mathbb{R}^d$, which corresponds to the emergence of a (conjectured) Goldstone boson. Similar results were also obtained for the Abelian gauge theory in four dimensions (see [**51, 66**]). Kennedy and King in [**70**] obtained a similar low temperature expansion for the Abelian Higgs model, which couples an XY model with a gauge fixing potential. Their proofs rely on a different approach, via a transformation introduced by [**14**] and a polymer expansion.

It is also of much interest to justify the spin wave conjecture separately for the longitudinal and transversal two-point functions of the rotator models, i.e., observables of the form $\langle \cos\theta(0)\cos\theta(x)\rangle_{\mu_\beta^{XY}}$ and $\langle \sin\theta(0)\sin\theta(x)\rangle_{\mu_\beta^{XY}}$. The best known result is due to Bricmont, Fontaine, Lebowitz, Lieb, and Spencer [**26**], where, relies on a combination of the infrared bound [**48**], a Mermin-Wagner type argument, and correlation inequalities, they perform a low temperature expansion of the truncated correlation function of the XY model and obtain the following expansion.

***Proposition 1.1.2* (Bricmont, Fontaine, Lebowitz, Lieb, and Spencer [26]). —** *There exist an inverse temperature $\beta_1 < \infty$ and two constants $c_1 > c_2 > 0$ such that, for any $\beta \geq \beta_1$,*

$$\frac{c_2}{\beta |x|^{d-2}} \leq \langle \sin\theta(0)\sin\theta(x)\rangle_{\mu_\beta^{XY}} \leq \frac{c_1}{\beta |x|^{d-2}}.$$

Despite these considerable progress, the rigorous derivations of the spin wave Conjecture (1.1.8) remain largely open. The main result of our paper, stated below, identifies the next-order term for the Villain model in dimensions three and higher, by obtaining the precise asymptotics of the two-point functions at low temperature.

***Theorem 1.* —** *For any dimension $d \geq 3$, there exist $\beta_0 = \beta_0(d)$ and $\alpha = \alpha(d) > 0$ such that, for any $\beta \geq \beta_0$, there exist constants $c_0 = c_0(\beta, d), c_1 = c_1(\beta, d)$, $c_2 = c_2(\beta, d)$, and such that, for all $\beta > \beta_0$, the transversal two-point function has the asymptotics*

$$\langle \sin\theta(0)\sin\theta(x)\rangle_{\mu_\beta^V} = \frac{c_2}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-2+\alpha}}\right), \tag{1.1.9}$$

and the spin-spin correlation function satisfies

$$\langle S_0 \cdot S_x \rangle_{\mu_\beta^V} = c_0 + \frac{c_1}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-2+\alpha}}\right). \tag{1.1.10}$$

Remark 1.1.3. — The proof of Theorem 1 yields the following characterization for the constant c_0

$$c_0 = \langle S_0 \rangle^2_{\mu_\beta^V}.$$

Regarding the constants c_1 and c_2, the free field computation (1.1.5) indicates that they should be close to the constant

$$C = -\frac{1}{\beta} \exp\left(G(0)/\beta\right) \frac{\Gamma(d/2-1)}{4\pi^{d/2}},$$

where Γ is the standard Gamma function. The constant C is defined so as to satisfy

$$\left\langle e^{i(\phi(0)-\phi(x))} \right\rangle_{\mu_\beta^{GFF}} = \exp\left(\frac{1}{\beta}(G(0)-G(x))\right) = \exp\left(G(0)/\beta\right) + \frac{C}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-1}}\right).$$

In this direction, the proof of Theorem 1 yields the identities

$$c_1 = C + O(e^{-c\beta}) \quad \text{and} \quad c_2 = -C + O(e^{-c\beta}).$$

Remark 1.1.4. — It follows from (1.1.9) and (1.1.10) that the two-point correlation function is asymptotically rotation invariant. Indeed, the proof yields rotation invariance for the Villain Gibbs measures that are invariant under the $\pi/2$-degree rotations and the reflections of the lattice. For more general Villain models, i.e., replacing the potential (1.1.3) by

$$v_{\beta,x,y}(\theta) := \sum_{m\in\mathbb{Z}} \exp\left(-\frac{\beta J_{x,y}}{2}(\theta + 2\pi m)^2\right),$$

for strictly positive, nearest neighbor and periodic coupling constants $J_{x,y}$, one expects the second order term to take the form of a more general $(2-d)$-homogeneous function.

We remark here that an alternative approach, based on elaborate renormalization group analysis, was developed in a series of works of Balaban, and culminated in **[15]**. They studied a class of Euclidean field theories that are invariant under the $O(N)$ symmetry group, for $N \geq 2$, and obtained results similar to Theorem 1 for these models.

We conclude the introduction by mentioning two open questions. The Gaussian spin wave approximation predicts that the two-point function of the XY model in $d \geq 3$ also admits a low temperature expansion like that stated in Theorem 1. The main challenge is a technical one: in the first step of the proof (described in Section 1.2 below), a duality transformation and a cluster expansion step are used to prove that the model can be expressed as gradient model with a strictly convex potential (this part of the proof follows well-known arguments **[51, 16]**). The specific structure of the Hamiltonian of the Villain model (1.1.2) allows an exact factorisation (in particular, the two-point function can be factorized as a Gaussian contribution and a vortex contribution, see Section 3, (3.1.6)). Such an exact factorization does not hold for the XY model and a new idea for renormalization is required to implement the argument.

The spin wave conjecture and the asymptotic two-point function (1.1.7) remains open for the XY and Villain model in $d = 2$. The renormalization argument developed by Falco [**45, 46**] does not directly apply, because by applying a duality transform to the XY and Villain model, one obtains a lattice Coulomb gas with infinite activity (instead of small activity). Building new insights into the renormalization group analysis, Bauerschmidt, Park and Rodriguez showed recently that the scaling limit of the two-dimensional Discrete Gaussian at high temperature is a continuous Gaussian free field (with an effective inverse temperature) in [**17, 18**]. Their result makes another progress toward the spin-wave conjecture for the two-dimensional Villain model in the low temperature regime ($\beta \gg 1$). Resolving the conjecture requires extending the results of [**17, 18**] to more singular test functions.

1.2. Strategy of the proof

We initiate a renormalization-Helffer-Sjöstrand-homogenization program to prove Theorem 1. The periodic potential of the XY and Villain model makes the interaction highly non-convex, and poses significant challenges to study their large scale behavior. Indeed, the ground states at zero temperature already leads to highly nontrivial variational problems (see, e.g., [**5**]). To overcome these difficulties, we start from the insight of Fröhlich and Spencer [**51**] (see also [**16**, Section 5]), applying a duality transformation and a cluster expansion to the Villain Gibbs measure. In the low temperature regime ($\beta \gg 1$), this argument shows that two-point function can be expressed as a non-linear and non-local observable of a uniformly convex gradient model (or uniformly convex $\nabla\phi$ model). Contrary to the Villain and XY models, tools from PDE and homogenization theory can be applied to study the behavior over large-scales of the uniformly convex $\nabla\phi$ model (see Section 1.2.2), which can thus be used to study the Villain model via the duality transformation of [**51**]. The general strategy described above encounters two difficulties. Firstly the convex model is not nearest neighbor, and has an infinite-range with exponential tail. Secondly the two-point function of the Villain model is mapped via the duality transform to a non-linear and non-local observable (see Proposition 3.1.1). Understanding the behavior of this observable requires a precise, quantitative theory to describe the large-scale behavior of the convex gradient model.

This first part of the proof thus consists of applying a duality transformation and cluster expansion to relate the Villain model to a uniformly convex $\nabla\phi$ model. It is the subject of Section 3 and mostly follows [**51**] and [**16**, Section 5]. The second part of the proof consists of studying quantitatively the large-scale behavior of the convex gradient Gibbs model and treating the non-linear, non-local observable arising from the arguments of [**51**] and [**16**, Section 5], and is the subject of the remaining sections.

One of the main tools to study $\nabla\phi$ model is the so-called Helffer-Sjöstrand equation, originally introduced by Helffer and Sjöstrand [**68**], Naddaf and Spencer [**82**] and

Giacomin, Olla and Spohn [**55**] to identify the scaling limit of the model. The main insight of [**68, 82**] is that the large-scale behavior of the $\nabla\phi$ model is related to the large-scale behavior of the solutions of an infinite-dimensional elliptic equation called the Helffer-Sjöstrand equation. The crucial observation of [**82**] is that the large-scale behavior of these solutions can be studied using techniques of homogenization.

At a high level, the proof of Theorem 1 consists of developing a quantitative homogenization theory for the Helffer-Sjöstrand equation and exploits the insights of the following three works: the work of Naddaf and Spencer [**82**], that relates large-scale behavior of the convex gradient Gibbs measure to an elliptic homogenization problem for the Helffer-Sjöstrand equation; the quantitative theory for homogenization by Armstrong, Kuusi and Mourrat [**7, 6**]; and the application of quantitative homogenization to the $\nabla\phi$ model by Armstrong and Wu [**8**]. However there is a distinct difference of our method compared to [**82, 7, 8**]. Firstly, the results of [**82**] are qualitative, and a quantitative theory is required to understand the behavior of the Villain model. To obtain a quantitative rate of homogenization it is crucial to have some decorrelation of the underlying random field. In [**7**], a straightforward mixing condition of the coefficient field is assumed. The argument in [**8**] relies on couplings based on the probabilistic interpretation of the equation to obtain decorrelation of the gradient field. In the present paper, we rely on the observation that this information can be obtained by studying another infinite-dimensional equation, the *second-order Helffer-Sjöstrand equation* (see [**29**, (2.12)] or Section 1.2.4); in particular, the decorrelation is a consequence of the decay estimates for the Green's function associated with the second-order Helffer-Sjöstrand operator. We note that the second-order equation appears in the work [**29**], and is closely related to techniques used to develop a quantitative theory of stochastic homogenization in [**61, 62, 59, 60**].

The following subsections provide a more detailed outline of the argument.

1.2.1. Sine-Gordon representation and polymer expansion. — The spin wave computation (1.1.8) is only heuristic and does not give the correct constants C_1, C_2. The main problem for the spin wave heuristics (1.1.8) is that it ignores the formation of *vortices*, which are defined on the faces of $\mathbb{Z}^d$. Kosterlitz and Thouless [**72**] gave a heuristic argument, indicating that the vortices interact like a neutral Coulomb gas taking integer-valued charges.

Our proof of Theorem 1 starts from an insight of Fröhlich and Spencer [**51**], which makes this observation rigorous. In particular, the correlation function of the Villain model in $\mathbb{Z}^d$, $d \geq 3$ can be mapped, by duality, to a statistical mechanical model with integer-valued and locally neutral charges on discrete 2-forms $\Lambda^2(\mathbb{Z}^d)$, interacting with Coulomb potential (see Section 3.1). By performing a Fourier transform of this Gibbs measure with respect to the charge variable, we obtain a helpful random field representation of the Coulomb gas, known as the sine-Gordon representation (see e.g., [**49, 50**]). When the temperature is low enough, opposite charges tend to bind together into neutral (short range) dipoles, therefore on large scales this Coulomb gas behaves like an effective dipole gas with a reduced effective activity of the charges.

This can be formalized by applying a one-step renormalization argument and a cluster expansion, following the presentation of [**16**, Chapter 5]. The renormalized Gibbs measure (see (3.1.16)) is a vector-valued random interface model in $\Lambda^2(\mathbb{Z}^d)$ with infinite range and uniformly convex potential. The question of the asymptotic behavior of the Villain correlation function is thus reduced to the question of the quantitative understanding of the large-scale properties of the random interface model.

1.2.2. Random surfaces and Helffer-Sjöstrand equation. — Our study of the large-scale properties of the random interface model starts from the insight of Naddaf and Spencer [**82**] that the fluctuations of the field are closely related to an elliptic homogenization problem for the Helffer-Sjöstrand equation [**68, 89**]. This approach has been used by Giacomin, Olla and Spohn in [**55**] to prove that the large-scale space-time fluctuations of the field is described by an infinite-dimensional Ornstein-Uhlenbeck process and by Deuschel, Giacomin and Ioffe to establish concentration properties and large deviation principles on the random surface (we also refer to [**88, 21, 22, 31, 30**] for extension of these results to some non-convex potentials, and [**73**] for a study of a more general class of Hamiltonians). The strategy presented in many of the aforementioned articles relies on a probabilistic approach: one can, through the Helffer-Sjöstrand representation, reduce the problem to a question of random walk in dynamic random environment, and then prove properties on this object, e.g., invariance principles, using the results of Kipnis and Varadhan [**71**], or annealed upper bounds on the heat kernel, using Delmotte and Deuschel [**41**]. However, the results obtained so far using this probabilistic approach are not quantitative. A more analytical approach was developed by Armstrong and Wu in [**8**], where they extend and quantify the homogenization argument of Naddaf and Spencer [**82**], resolved an open question posed by Funaki and Spohn [**53**] regarding the C^2 regularity of surface tension, and the fluctuation-dissipation conjecture of [**55**].

Besides the approach based on the Helffer-Sjöstrand equation and the random walk representation, various techniques have been successfully used on the model. Funaki and Spohn [**53**] established the hydrodynamic limit of the model relying on methods developed in the setting of the Ginzburg-Landau equation with a conserved order parameter [**65**]. A renormalization group approach has been implemented in the works of Adams, Kotecký, Müller [**3**] and Adams, Buchholz, Kotecký, Müller [**2**]. In these contributions, the authors study the $\nabla\phi$ model for a general class of (perturbative) *non-convex potentials* (in a low temperature regime) and establish (among other results) regularity properties as well as the strict convexity of the surface tension of the model. The articles [**3, 2**] differ from ours in various aspects. In [**3, 2**], the authors consider a nonconvex perturbation of Gaussian, and proved after successive renormalizations the surface tension (i.e., the log partition function under different tilts) gains sufficient regularity and convexity. In the present article, the gradient-type model obtained from the Villain model by duality is uniformly convex, and the main difficulty relies on the specific structure of the model: the Hamiltonian has infinite-range, the observable we wish to study is highly non-linear and non-local. Therefore it is not enough to prove the Gibbs measure converges to a Gaussian free field in the scaling limit, and we need to estimate the correlation of nonlinear functions of the field with high precision, which we do by implementing methods from PDE and homogenization theory.

On a high level, we follow the analytical approach, namely the program developed in [**82, 8**] on homogenization for the random interface models. Since the sine-Gordon representation and the polymer expansion give a random interface model valued in the vector space $\mathbb{R}^{\binom{d}{2}}$ with long range and uniformly convex potential, an application of the strategy of Naddaf and Spencer [**82**] to this model leads to the Helffer-Sjöstrand operator

$$\mathscr{L} := -\Delta_\phi + \mathscr{L}_{\mathrm{spat}}, \tag{1.2.1}$$

which is an infinite-dimensional elliptic operator acting on functions defined in the space $\Omega \times \mathbb{Z}^d$ where Ω is the set of functions $\phi : \mathbb{Z}^d \to \mathbb{R}^{\binom{d}{2}}$ (see (3.4.4) for the precise definition of this operator), where Ω is the space of functions from $\mathbb{Z}^d$ to $\mathbb{R}^{\binom{d}{2}}$ in which the vector-valued random interface considered in this article takes its values. The operator Δ_ϕ is the (infinite-dimensional) Laplacian computing derivatives with respect to the height of the random surface and $\mathscr{L}$ is an operator associated with a uniformly elliptic *system of equations* with infinite range (and with exponential decay on the size of the long range coefficients) on the discrete lattice $\mathbb{Z}^d$. The analysis of these systems requires to overcome some difficulties; a number of properties which are valid for elliptic equations, and used to study the random interface models, are known to be false for elliptic systems. It is for instance the case for the maximum principle, which is used to obtain a random walk representation, the De Giorgi-Nash-Moser regularity theory for uniformly elliptic and parabolic PDE (see [**84, 39**], [**57**, Section 8] and the counterexample of De Giorgi [**40**]) and the Nash-Aronson estimate on the heat kernel (see [**11**]).

To resolve this lack of regularity, we rely on a perturbative argument, and make use of ideas from *Schauder theory* (see [**67**, Section 3]), as well as the ones from *the large-scale regularity in homogenization* (see Avellaneda, Lin [**12, 13**] and Armstrong, Smart [**10**]); we leverage on the fact that the inverse temperature β is chosen very large so that the elliptic operator $\mathscr{L}$ can be written

$$\mathscr{L}_{\mathrm{spat}} := -\frac{1}{2\beta}\Delta + \mathscr{L}_{\mathrm{pert}},$$

where the operator $\mathscr{L}_{\mathrm{pert}}$ is a perturbative term; its typical size is of order $\beta^{-\frac{3}{2}} \ll \beta^{-1}$. One can thus prove that any solution u of the Equation (1.2.1) is well-approximated on every scale by a solution $\overline{u}$ of the equation $-\Delta_\phi - \frac{1}{2\beta}\Delta$ for which the regularity can be easily established. It is then possible to borrow the strong regularity properties of the function $\overline{u}$ and transfer it to the solution of (1.2.1). This strategy is implemented in Section 5 and allows us to prove the $C^{0,1-\varepsilon}$-regularity of the solution of the Helffer-Sjöstrand equation, and to deduce from this regularity property various estimates on other quantities of interest (e.g, decay estimates on the heat kernel in dynamic random environment, decay and regularity for the Green's matrix associated with the Helffer-Sjöstrand operator). The regularity exponent ε depends on the dimension d and the inverse temperature β, and tends to 0 as β tends to infinity; in the perturbative regime, the result turns out to be much stronger than the $C^{0,\alpha}$-regularity provided by the De Giorgi-Nash-Moser theory (for some tiny exponent $\alpha > 0$) in the case of elliptic equations, and allows to quantify (precisely) the mixing properties of the random field.

1.2.3. Stochastic homogenization. — The main difficulty to establish Theorem 1 is that since the Villain model is not exactly solvable, the dependence of the constants c_1 and c_2 on the dimension d and the inverse temperature β is highly non explicit; one does not expect to have a simple formula for these coefficients. However, it is necessary to analyze them in order to prove the expansions (1.1.9) and (1.1.10); this is achieved by using tools from the quantitative theory of stochastic homogenization.

This theory is typically interested in the understanding of the large-scale behavior of the solutions of the elliptic equation

$$-\nabla \cdot \mathbf{a}(x)\nabla u = 0 \text{ in } \mathbb{R}^d, \tag{1.2.2}$$

where $\mathbf{a}$ is a random, uniformly elliptic coefficient field that is stationary and ergodic. The general objective is to prove that, on large scales, the solutions of (1.2.2) behave like the solutions of the elliptic equation

$$-\nabla \cdot \bar{\mathbf{a}}\nabla u = 0 \text{ in } \mathbb{R}^d, \tag{1.2.3}$$

where $\bar{\mathbf{a}}$ is a constant uniformly elliptic coefficient called *the homogenized matrix.* The theory was initially developed in the 80's, in the works of Kozlov [**74**], Papanicolaou and Varadhan [**86**], and Yurinskiĭ [**91**]. Dal Maso and Modica [**32, 33**] extended these results a few years later to non-linear equations using variational arguments inspired by Γ-convergence. All of these results rely on the ergodic theorem, and are therefore purely qualitative.

The main difficulty in the establishment of a quantitative theory is to transfer the quantitative ergodicity encoded in the coefficient field $\mathbf{a}$ to the solutions of the equation. This problem was addressed in a satisfactory fashion for the first time by Gloria and Otto in [**61, 62**], where, building upon the ideas of [**83**], they used spectral gap inequalities (or concentration inequalities) to transfer the quantitative ergodicity of the coefficient field to the solutions of (1.2.2). These results were then further developed in [**64, 63, 59, 60**].

Another approach, which is the one pursued in this article, was initiated by Armstrong and Smart in [**10**], who extended the techniques of Avellaneda and Lin [**12, 13**], the ones of Dal Maso and Modica [**32, 33**] and obtained an algebraic, suboptimal rate of convergence for the homogenization error of the Dirichlet problem associated with the non-linear version of the Equation (1.2.2). These results were then improved in [**9, 6, 7**] to obtain optimal rates. Their approach relies on mixing conditions on the coefficient fields and on the quantification of the closeness of dual monotone quantities (see Section 6). An extension of the techniques of [**7**] to the setting of differential forms (which also appear in this article in the dual Villain model) can be found in [**35**], and to the uniformly convex gradient field model in [**34**]. In [**81**], Mourrat and Otto study the correlation structure of the corrector and prove that it is similar, in the large-scale limit, to the one of a variant of a Gaussian free field. Their strategy shares some similarities with ours: under some suitable assumptions on the coefficient field, they use a Helffer-Sjöstrand representation formula to study the correlation of the corrector, and reduce the problem to the question of the quantitative homogenization of the Green's function associated with the heterogeneous operator (1.2.2).

To prove Theorem 1, we apply the techniques of [**7**] to the Helffer-Sjöstrand equation to prove the quantitative homogenization of the mixed derivative of the Green's matrix associated with this operator. The strategy can be decomposed into two steps.

The first one relies on the variational structure of the Helffer-Sjöstrand operator and is the main subject of Section 6: following the arguments of [**7**, Section 2], we define two subadditive quantities, denoted by ν and ν^*. The first one corresponds to the energy of the Dirichlet problem associated with the Helffer-Sjöstrand operator (1.2.1) in a domain $U \subseteq \mathbb{Z}^d$ and subject to affine boundary condition, the second one corresponds to the energy of the Neumann problem of the same operator with an affine flux. Each of these two quantities depends on two parameters: the domain of integration U and the slope of the affine boundary condition, denoted by p (for ν) and p^* (for ν^*). These energies are quadratic, uniformly convex with respect to the variables p and p^*, and are approximately convex dual to one another. They additionally satisfy a subadditivity property with respect to the domain U, and one can show that they converge as the size of the domain tends to infinity to a pair of quadratic, convex dual functions, i.e., there exists a positive definite matrix $\bar{\mathbf{a}}$ such that

$$\nu\left(U,p\right) \underset{|U|\to\infty}{\longrightarrow} \frac{1}{2}p\cdot\bar{\mathbf{a}}p \quad \text{and} \quad \nu^*\left(U,p^*\right) \underset{|U|\to\infty}{\longrightarrow} \frac{1}{2}p^*\cdot\bar{\mathbf{a}}^{-1}p^*.$$

The matrix $\bar{\mathbf{a}}$ plays a similar role as the homogenized matrix in (1.2.3); in the case of the present random interface model, it gives the covariance matrix of the continuous (homogenized) Gaussian free field which describes the large-scale behavior of the random surface as established in [**82**]. The objective of the proofs of Section 6 is to quantify this convergence and to obtain an algebraic rate: we show that, for large β, there exists an exponent $\alpha > 0$ depending only on the dimension d such that for any cube $\square \subseteq \mathbb{Z}^d$ of size $R > 0$,

$$\left|\nu\left(\square,p\right) - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right| + \left|\nu\left(\square,p^*\right) - \frac{1}{2}p^*\cdot\bar{\mathbf{a}}^{-1}p^*\right| \leq CR^{-\alpha}. \tag{1.2.4}$$

The strategy to prove the quantitative rate (1.2.4) relies on the approximate convex duality of the maps $p \mapsto \nu\left(U,p\right)$ and $p^* \mapsto \nu^*\left(U,p^*\right)$. Following [**7**], we use a multiscale argument to prove that, as one passes to a larger scale, the *convex duality defect*

$$p \mapsto \inf_{p^*\in\mathbb{R}^d}\left[\nu\left(\square,p\right) + \nu^*\left(\square,p^*\right) - p\cdot p^*\right],$$

must contract by a multiplicative factor strictly smaller than 1, and thus it is equal to 0 in the infinite volume limit. More precisely we show that the convex duality defect can be controlled by the subadditivity defect, and then iterate the result over all the scales from 1 to R to obtain (1.2.4) (see Section 6.1.3). As a byproduct of the proof, we obtain a quantitative control on the sublinearity of the finite-volume corrector defined as the solution of the Dirichlet problem: given an affine function l_p of slope p, and a cube $\square_R := [-R, R]^d \cap \mathbb{Z}^d$ of size R,

$$\begin{cases} \mathscr{L}\left(l_p + \chi_{R,p}\right) = 0 \text{ in } \square_R \times \Omega, \\ \qquad\qquad \chi_{R,p} = 0 \text{ on } \partial\square_R \times \Omega. \end{cases}$$

This estimate takes the following form

$$\|\chi_{R,p}\|_{\underline{L}^2(\square_R,\mu_\beta)} \leq \frac{C}{R^{1-\alpha}}, \tag{1.2.5}$$

where the average L^2-norm is considered over both the spatial variable and the random field (see (2.1.5)).

The second step in the argument, which extends the results of [**8**], is to prove quantitative homogenization of the mixed derivative of the Green's matrix associated with the Helffer-Sjöstrand operator (1.2.1); it is the subject of Section 7. In the setting of the divergence form elliptic operator (1.2.2), the properties of the Green's function are well-understood: moment bounds on the Green's function, its gradient and mixed derivative are proved in [**41, 19, 28**], and quantitative homogenization estimates are proved in [**7**, Sections 8 and 9] and in [**20**]. The argument used here relies on a common strategy in stochastic homogenization: the two-scale expansion. It is implemented as follows: the large-scale behavior of the fundamental solution $\mathscr{G} : \Omega \times \mathbb{Z}^d \to \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}}$ of the elliptic system

$$\mathscr{L}\mathscr{G} = \delta_0 \text{ in } \mathbb{Z}^d \times \Omega,$$

is described by the (deterministic) fundamental solution $\overline{G} : \mathbb{Z}^d \to \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}}$ of the homogenized elliptic system

$$-\nabla \cdot \bar{\mathbf{a}} \nabla \overline{G} = \delta_0 \text{ in } \mathbb{Z}^d.$$

The proof of this result relies on a two-scale expansion for systems of equations: we select a suitable cube $\square \subseteq \mathbb{R}^d$ and define the function, for any $k \in \{1, \ldots, \binom{d}{2}\}$

$$\mathscr{H}_{\cdot k} := \overline{G}_{\cdot k} + \sum_{i=1}^{d} \sum_{j=1}^{\binom{d}{2}} \chi_{R,e_{ij}} \nabla_i \overline{G}_{jk}.$$

We then compute the value of $\mathscr{L}\mathscr{H}$ and prove, by using the quantitative information obtained on the corrector (1.2.5), that this value is small in a suitable functional space. This argument shows that the function $\mathscr{H}$ (resp. its gradient) is quantitatively close to the functions $\mathscr{G}$ (resp. its gradient). Once this is achieved, we can iterate the argument to obtain a quantitative homogenization result for the mixed derivative of the Green's matrix. The overall strategy is similar to the one in the case of the divergence form elliptic Equations (1.2.2) but a number of technicalities need to be treated along the way pertaining to either the Witten Laplacian Δ_ϕ (this difficulty has been successfully addressed in [**8**]), and the infinite range of the elliptic operator $\mathscr{L}_{\mathrm{spat}}$ (using the exponential decay of the interaction is enough to adapt the arguments developed in the nearest-neighbor setting).

1.2.4. Second-order Helffer-Sjöstrand equation. — As we mentioned, the method pursued in this paper differs from [**7**] and [**8**] and is based on the regularity theory of the second-order Helffer-Sjöstrand equation. We note that, contrary to the case of the homogenization of the elliptic Equation (1.2.2), the subadditive quantities are deterministic objects and are applied to the operator (1.2.1) which is essentially infinite-dimensional. To quantify the subadditive ergodic theorem and obtain the rate of convergence (1.2.4), it is crucial that the random fields $\nabla\phi$ that appears in the definition of ν and ν^* decorrelates (see Definition 6.1.4). While the proofs of quantitative rate of convergence in [**7**, Section 2] rely on a finite range dependence assumption of the coefficient field, we rely here on the regularity properties of the Helffer-Sjöstrand operator to prove sufficient decorrelation estimates on the field. The same issues were addressed in the work of Armstrong and Wu [**8**], to study the $\nabla\phi$ model and prove C^2-regularity of the surface tension conjectured by Funaki and Spohn [**53**]; the arguments presented there are different as they rely on couplings based on the probabilistic interpretation of the equation to obtain sufficient decorrelation of the discrete gradient $\nabla\phi$. In the present paper, we rely on the observation of Conlon and Spencer [**29**] that if u is a solution to the Helffer-Sjöstrand Equation (1.2.1), then the derivative of the function u with respect to the field ϕ, i.e., the map $v : (x, \phi, y) \mapsto \partial_y u(x, \phi)$, for $x, y \in \mathbb{Z}^d$ and $\phi \in \Omega$, solves a second-order Helffer-Sjöstrand equation of the form

$$\Delta_\phi v(x,y,\phi) + \mathscr{L}_{\mathrm{spat},x} v(x,y,\phi) + \mathscr{L}_{\mathrm{spat},y} v(x,y,\phi) + (\partial_y \mathscr{L})\, v = 0 \text{ in } \mathbb{Z}^d \times \mathbb{Z}^d \times \Omega. \tag{1.2.6}$$

We refer to Section 5.4 for a precise definition. This operator is then used in [**29**] to obtain uniform third moment bounds for the $\nabla\phi$ Gibbs measure. We note that this strategy is very similar to the one developed in stochastic homogenization in [**61, 62, 59, 60**]. In this paper we exploit more precise information of the operator, and apply the $C^{0,1-\varepsilon}$ regularity theory to obtain decay estimates on the Green's function associated with (1.2.6). In particular, we obtain the regularity theory for the second-order Helffer-Sjöstrand operator for large β, namely, the off-diagonal decay of the associated Green's matrix, its gradient, and its mixed derivative (see Corollary 5.4.4). These properties can be used to quantify the ergodicity of the Helffer-Sjöstrand equation and obtain the quantitative rate of convergence (1.2.4).

The second-order Helffer-Sjöstrand equation also plays a crucial role to derive Theorem 1 from the homogenization results. Applying the duality, we map the two-point function of the Villain model to a non-local observable (see Proposition 3.1.1). This non-local observable is then analyzed by repeated applications of the Helffer-Sjöstrand representation to single out the main contribution (thus the second-order Helffer-Sjöstrand operator emerges), and the $C^{0,1-\varepsilon}$ regularity theory is crucially applied to control the remainder terms (see Section 4.4 and Section 4.5 for the details).

1.2.5. First order expansion of the two-point functions. — The first order expansion of the two-point function stated in Theorem 1 is obtained by post-processing all the arguments above. We first use the sine-Gordon representation and the polymer expansion to reduce the question to the understanding of the large scale behavior of a vector-valued random surface model, whose Hamiltonian is a perturbation of the one of a Gaussian free field, and use the properties of the Helffer-Sjöstrand equation to treat the problem. The proof of Theorem 1 is decomposed into three parts:

- We establish a $C^{0,1-\varepsilon}$-regularity theory for the solutions of the Helffer-Sjöstrand and second-order Helffer-Sjöstrand operators by using the techniques of Schauder regularity (through a perturbative argument) in order to obtain a precise understanding of the correlation structure of the random field, this is done in Section 5;
- We prove a quantitative homogenization theorem for the mixed derivative associated with the Helffer-Sjöstrand operator (Theorem 2), this is done in Sections 6 and 7;
- We post-process the results of the two arguments above to prove Theorem 1. The proof relies on the study of the non-local observable introduced in Proposition 3.1.1; it requires to analyze a number of terms, to isolate the leading order terms, and to estimate quantitatively the lower order ones. It is rather technical and is split into two sections: in Section 4, we present a detailed sketch of the argument, isolate the leading order from the lower order terms, and state the estimates on each of these terms. Section 8 is devoted to the proof of the technical estimates.

1.3. Organization of the paper

This article is the short version of the v1 of arxiv preprint [**36**], which contains in addition some detailed but standard computations which are recalled here without a proof. In the next section, we introduce some preliminary notation and results. In Section 3, we recall the dual formulation of the Villain model in terms of a vector-valued random interface model, based on the ideas of Fröhlich and Spencer [**51**] and following the presentation of Bauerschmidt [**16**]. We then derive the Helffer-Sjöstrand equation for the renormalized measure and state the main regularity estimates on the Green's matrix proved in Section 5, and the quantitative homogenization of the mixed derivative of the Green's matrix proved in Sections 6 and 7. In Section 4, we sketch the proof of the main theorem, assuming the $C^{0,1-\varepsilon}$ regularity for the solutions of the Helffer-Sjöstrand equation (established in Section 5), and the quantitative homogenization of the mixed derivative of the Green's matrix (established in Sections 7 and 8). Finally in Section 8, we give detailed proofs of the claims in Section 4.

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