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## ON THE HAUSDORFF DIMENSION OF THE RAUZY GASKET

BY ARTUR AVILA, PASCAL HUBERT & ALEXANDRA SKRIPCHENKO

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ABSTRACT. — In this paper, we prove that the Hausdorff dimension of the Rauzy gasket is less than 2. By this result, we answer a question addressed by Pierre Arnoux. Also, this question is a very particular case of the conjecture stated by S.P. Novikov and A. Maltsev in 2003.

RÉSUMÉ (*Sur la dimension de Hausdorff de la baderne de Rauzy*)

Dans cet article, nous montrons que la dimension de Hausdorff de la baderne de Rauzy est strictement inférieure à 2. Ce résultat répond à une question de Pierre Arnoux. C'est aussi une réponse, dans un cas très particulier, à une conjecture posée par S.P. Novikov et A. Maltsev en 2003.

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## 1. Introduction

The Rauzy gasket (see Figure 1) was described for the first time by G. Levitt in 1993 in [12] and was associated with the simplest example of pseudogroups of rotations. Later the Rauzy gasket was studied by I. Dynnikov and R. De Leo (see [8]) in connection with Novikov's problem ([15]) of plane sections of triply periodic surfaces. In [1] independently P. Arnoux and S. Starosta reintroduced this object as the subset of standard 2-dimensional simplex associated with letter frequencies of ternary episturmian words. The name Rauzy gasket was used for the first time in [1].

The same fractal appears in connection with systems of isometries of thin type that are described by 2 independent parameters. A detailed description of the last approach is provided in the next section. In all these cases, the Rauzy gasket plays the role of a parameter space endowed with a dynamics by piecewise projective maps.

It was proved by Levitt and J.-C. Yoccoz in [12], Arnoux and Starosta in [1] and by Dynnikov and De Leo ([8]) by different techniques that the Rauzy gasket has zero Lebesgue measure (see, in particular, [14] for the main approach used by Arnoux and Starosta to prove their result).

Hausdorff dimension of the Rauzy gasket was estimated numerically in [8] (1.7 and 1.8 were suggested as lower and upper bounds). However, there were no theoretical estimates. Arnoux asked whether this Hausdorff dimension is less than 2 or equal to 2 (see also [1] for other interesting open questions). The same problem but for much more general situation was posed by A. Maltsev and S.P. Novikov in [13]. In this paper we prove:

**THEOREM 1.** — The Hausdorff dimension of the Rauzy gasket is less than 2.

**REMARK.** — Our statement also proves the conjecture about the Hausdorff dimension of the set of chaotic regimes formulated by A. Maltsev and S.P. Novikov that we mentioned above for a very particular family of surfaces. This result follows directly from our theorem and the construction in [8].

**REMARK.** — An upper bound can be deduced from the proof of Theorem 1 but it would be much weaker than the known numerical estimates.

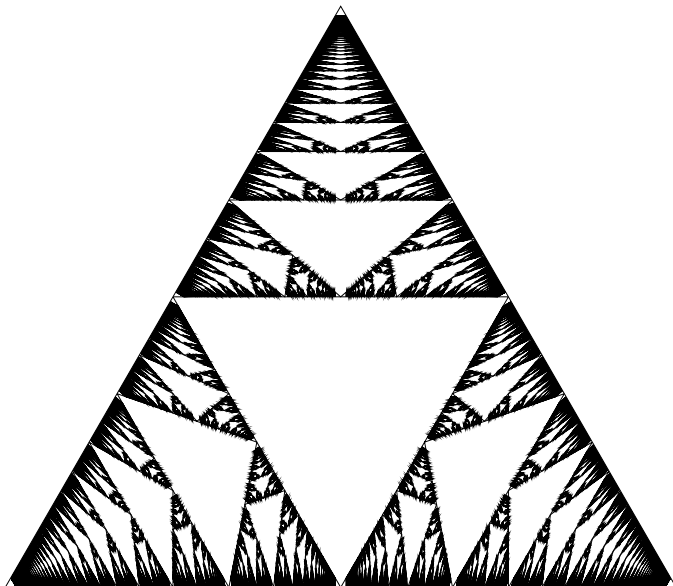


FIGURE 1. The Rauzy Gasket

**1.1. Organization of the paper.** — The paper is organized as follows.

In Section 2 we recall the definition of systems of isometries and describe the particular family we work with in the current paper.

In Section 3 we describe the Rauzy induction for systems of isometries and the related symbolic dynamics. In particular, we define the Rauzy gasket in terms of systems of isometries and describe the corresponding Markov map and Markov partition.

Section 4 is dedicated to the cocycle associated with the induction. Like in case of IET, the definition requires a suspension construction that is also presented in the same section. This cocycle will be used later for the construction of the suspension flow.

In Section 5 we prove that the Markov map is uniformly expanding in a sense of [3].

In 6 and in 7 we provide some distortion estimates for the cocycle that are based on so called Kerckhoff lemma (see Appendix A in [3] and Theorem 4.2 in [4]).

Section 8 is about the roof function and the suspension flow: we construct the roof function associated with the cocycle and use this roof function to define the flow.