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J. FRAHM

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Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
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Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
memoires@smf.emath.fr • http://smf.emath.fr/

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Jan Frahm

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J. Frahm
Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus, Denmark.
E-mail : `frahm@math.au.dk`

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CONFORMALLY INVARIANT DIFFERENTIAL OPERATORS ON HEISENBERG GROUPS AND MINIMAL REPRESENTATIONS

Jan Frahm

Abstract. – For a simple real Lie group G with Heisenberg parabolic subgroup P, we study the corresponding degenerate principal series representations. For a certain induction parameter the kernel of the conformally invariant system of second order differential operators constructed by Barchini, Kable and Zierau is a subrepresentation which turns out to be the minimal representation. To study this subrepresentation, we take the Heisenberg group Fourier transform in the non-compact picture and show that it yields a new realization of the minimal representation on a space of L^2-functions. The Lie algebra action is given by differential operators of order ≤ 3 and we find explicit formulas for the functions constituting the lowest K-type.

These L^2-models were previously known for the groups $\mathrm{SO}(n,n)$, $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ by Kazhdan and Savin, for the group $G_{2(2)}$ by Gelfand, and for the group $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ by Torasso, using different methods. Our new approach provides a uniform and systematic treatment of these cases and also constructs new L^2-models for $E_{6(2)}$, $E_{7(-5)}$ and $E_{8(-24)}$ for which the minimal representation is a continuation of the quaternionic discrete series, and for the groups $\widetilde{\mathrm{SO}}(p,q)$ with either $p \geq q = 3$ or $p, q \geq 4$ and $p+q$ even.

As a byproduct of our construction, we find an explicit formula for the group action of a non-trivial Weyl group element that, together with the simple action of a parabolic subgroup, generates G.

Résumé **(Opérateurs différentiels conformément invariants pour des groupes d'Heisenberg et représentations minimales)**

Pour un groupe de Lie réel simple G, ayant pour sous-groupe parabolique de Heisenberg P, nous étudions les représentations de la série principale dégénérée associées à ces données. La représentation minimale peut être identifiée au noyau du système d'opérateurs différentiels conformément invariants construit par Barchini, Kable et Zierau, pour un paramètre d'induction convenable. Pour étudier cette représentation,

nous utilisons la transformation de Fourier pour le groupe d'Heisenberg dans la réalisation non-compacte et nous prouvons que cela conduit à une nouvelle réalisation de la représentation minimale sur un espace de fonctions L^2. L'action de l'algèbre de Lie est donnée par des opérateurs différentiels d'ordre ≤ 3 et nous trouvons des formules explicites pour les fonctions réalisant les K-types minimaux.

Ces modèles L^2 étaient construits pour les groupes $\mathrm{SO}(n,n)$, $E_{6(6)}$, $E_{7(7)}$ et $E_{8(8)}$ par Kazhdan et Savin, pour le groupe $G_{2(2)}$ par Gelfand, et pour le groupe $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ par Torasso, en utilisant différentes méthodes. Notre nouvelle approche fournit un traitement uniforme et systèmatique de ces exemples et construit également des nouveaux modèles L^2 pour $E_{6(2)}$, $E_{7(-5)}$ et $E_{8(-24)}$, pour lesquels la représentation minimale est un prolongement de la série discrète quaternionique, ainsi que pour les groupes $\widetilde{\mathrm{SO}}(p,q)$ pour $p \geq q = 3$ ou pour $p, q \geq 4$ et $p + q$ pair.

Comme conséquence de notre construction, nous trouvons une formule explicite pour l'action d'un élément non trivial du groupe de Weyl qui, en addition à l'action simple d'un sous-groupe parabolique, génère le groupe G.

CONTENTS

INTRODUCTION

The classification of all irreducible unitary representations of a semisimple Lie group is one of the key problems in representation theory, and it is still unsolved for most groups. A guiding principle for the classification is the orbit philosophy which proposes a tight relation between the unitary dual of a semisimple group G and the set of coadjoint orbits in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. For elliptic and hyperbolic coadjoint orbits, cohomological and parabolic induction provide explicit constructions of the corresponding unitary representations, and the resulting representations make up a large part of the unitary dual. For nilpotent orbits, however, it is not clear in general how to apply the orbit philosophy to construct unitary representations. Among the finitely many nilpotent coadjoint orbits of a simple Lie group, there are one or two of minimal dimension, depending on whether the group is of Hermitian type or not. Irreducible unitary representations corresponding to a minimal nilpotent coadjoint orbit are called *minimal representations*. They are often unique and in general a group can only have finitely many equivalence classes of minimal representations.

The most prominent example of a minimal representation is the metaplectic representation (also referred to as oscillator or Segal-Shale-Weil representation) of the metaplectic group $\mathrm{Mp}(n,\mathbb{R})$, a double cover of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ (see **[84]** for Weil's original work). We refer the reader to Folland's book **[13]** for a detailed account on the construction of this representation and some of its properties. Although the metaplectic representation plays an important role within the representation theory of the metaplectic group, it is mostly its relevance in other areas of mathematics and physics that have made this particular representation a truly fascinating object within the last few decades.

The key role of the metaplectic representation in the representation theory of real reductive groups is in the context of the theta correspondence, also referred to as Howe's dual pair correspondence. This correspondence was defined by Howe **[39]** and relates irreducible representations of two different groups G_1 and G_2 that occur inside the metaplectic group $\mathrm{Mp}(n,\mathbb{R})$ as a so-called *dual pair*, i.e., G_1 and G_2 are mutual centralizers of each other. The theta correspondence is a map that associates to a representation π of G_1 occurring inside the metaplectic representation a representation $\theta(\pi)$ of G_2 such that $\pi\otimes\theta(\pi)$ occurs as a quotient of the metaplectic representation restricted to $G_1 \times G_2$. It was shown by Howe **[40]** that this establishes a bijection between certain irreducible representations of G_1 and G_2.

The metaplectic representation also has a version over non-Archimedean local fields, over global fields and even over finite fields. Corresponding theta correspondences were established by Waldspurger **[83]**, Mínguez **[66]**, Gan-Takeda **[20]** and Gan-Sun **[19]** for the case of local non-Archimedean fields, and by Rallis **[70]** for global fields, and this is still a very active line of research. For instance, the theta correspondence has been applied to the construction of new representations, to classification problems, and to branching problems.

Apart from the theta correspondence, the metaplectic representation has also been used in various other contexts such as classical invariant theory **[40, 41]**, theta series and the Maslov index **[63]**, the Siegel-Weil formula in automorphic forms **[16, 60, 85]**, harmonic analysis **[13, 30]** and quantum mechanics **[88]**, just to mention a few.

It seems natural to try to extend the extremely rich theory of the metaplectic representation as a minimal representation of the metaplectic group $\mathrm{Mp}(n, \mathbb{R})$ to minimal representations of more general reductive groups. The first step in this program is the construction and classification of all minimal representations. For reductive groups over Archimedean local fields, many different constructions can be found in the literature. Let us list a few of them without claiming to be complete: Brylinski-Kostant **[5]**, Binegar-Zierau **[4]**, Dvorsky-Sahi **[9]**, Gelfand **[21]**, Gross-Wallach **[29]**, Hilgert-Kobayashi-Möllers **[38]**, Kazhdan-Savin **[49]**, Kobayashi–Ørsted **[55]**, Li **[62]**, Möllers-Schwarz **[68]**, Sabourin **[72]**, Savin **[73]**, Torasso **[78, 79]**, Vogan **[81, 82]**. It was believed that the thus obtained list of minimal representations is in fact exhaustive, and this was recently shown by Tamori **[77]**, building on ideas of Gan-Savin **[18]**. For p-adic groups, minimal representations were constructed by Kazhdan-Savin **[49]**, Rumelhart **[71]**, Savin **[73]**, Torasso **[79]** and Weissman **[86]**. An overview together with a global picture can be found in the paper by Gan and Savin **[18]**.

Subsequently, minimal representations of general reductive groups over local fields have turned out to be useful from various different points of view. For instance, their geometric realizations are a particularly rich source to do classical harmonic analysis, a viewpoint that has been particularly advocated by Kobayashi **[52]**. Special functions such as orthogonal polynomials or Bessel functions are often used to express explicit K-finite vectors in the representation spaces, see e.g., **[37]**. Moreover, explicit geometric realizations are often connected to interesting analytic and geometric problems such as partial differential equations on manifolds, see **[54, 55, 56]**. Many of these features also become apparent in this work.

Minimal representations can further be used to study certain Fourier coefficients of automorphic forms **[3, 26, 25, 48]** and to construct more general theta series, see **[48]**. Finally, versions of the theta correspondence have been established in some cases involving exceptional groups, see e.g., the work of Ginzburg-Jiang **[22]**, Ginzburg-Rallis-Soudry **[23]**, Gross-Savin **[27]**, Huang-Pandzic-Savin **[42]**, Li **[61, 62]**, Loke-Savin **[64]**, Magaard-Savin **[65]** and Weissman **[87]**, just to name a few.

What many of these applications have in common is that they rely on a rather explicit realization of the minimal representation. In particular in the Archimedean setting, where delicate analytic problems arise, it is often vital to be able to construct explicit vectors in the representation spaces and compute group actions on them. There is one particular type of realization that has turned out to be extremely useful for such purposes, a realization referred to as *L^2-model* or *Schrödinger model*. In the Archimedean context, one could define an L^2-model of a unitary representation as a realization on a Hilbert space of L^2-functions such that the Lie algebra acts by differential operators. L^2-models of minimal representations have essentially been constructed in two different settings which we now briefly describe.

The first class of groups for which L^2-models of minimal representations have been constructed in a uniform way are simple Lie groups G possessing a parabolic subgroup $P = MAN$ with abelian nilradical N, also referred to as *Siegel parabolic subgroups*. For such groups G, minimal representations can often be found as proper subrepresentations of the corresponding degenerate principal series $\mathrm{Ind}_P^G(\chi)$ for a certain real-valued character χ of P. The subrepresentations arise as the kernel of a system of second order differential operators. These differential operators can most easily be described in the so-called *non-compact picture* of the degenerate principal series. The non-compact picture is a realization on a space of functions on the opposite nilradical $\overline{N}$. Identifying $\overline{N}$ with its Lie algebra $\overline{\mathfrak{n}}$, the differential operators become second order constant coefficient differential operators on $\overline{\mathfrak{n}}$. For instance, for $G = \mathrm{O}(p,q)$ we have $\overline{\mathfrak{n}} \simeq \mathbb{R}^{p+q-2}$ and the system consists of a single differential operator whose symbol is a quadratic form of signature $(p-1, q-1)$ (see [**54**]), and for $G = \mathrm{Mp}(n,\mathbb{R})$ the system of differential operators on $\overline{\mathfrak{n}} \simeq \mathrm{Sym}(n,\mathbb{R})$, the real symmetric $n \times n$ matrices, has as symbols the 2×2 minors.

In order to obtain an L^2-model of this subrepresentation, the Euclidean Fourier transform $\mathcal{S}'(\overline{\mathfrak{n}}) \to \mathcal{S}'(\mathfrak{n})$, $u \mapsto \widehat{u}$, is employed. It turns the system of constant coefficient differential operators $P(\partial)$ into a system of multiplication operators $P(x)$ and hence the differential equations $P(\partial)u = 0$ turn into $P(x)\widehat{u} = 0$ which implies a support condition $\operatorname{supp}\widehat{u} \subseteq \{P(x) = 0\}$ on $\mathfrak{n}$. For instance, for $G = \mathrm{O}(p,q)$ the Fourier transform of a function in the subrepresentation is a distribution supported on the isotropic cone in $\mathfrak{n} \simeq \mathbb{R}^{p+q-2}$ associated with a quadratic form of signature $(p-1, q-1)$, and for $G = \mathrm{Mp}(n,\mathbb{R})$ they are supported on the zero set of all 2×2 minors in $\mathfrak{n} \simeq \mathrm{Sym}(n,\mathbb{R})$ which is the subvariety of rank one symmetric $n \times n$ matrices. In general, the submanifold on which these Fourier transforms are supported is an orbit $\mathcal{O} \subseteq \mathfrak{n}$ of $\mathrm{Ad}(MA)$ of minimal possible dimension, and one can show that in many cases this yields a realization of the minimal representation on $L^2(\mathcal{O}, d\mu)$ with respect to a certain $\mathrm{Ad}(MA)$-equivariant measure $d\mu$ on $\mathcal{O}$. This was first observed by Vergne-Rossi [**80**] in the case of Hermitian groups and later generalized by Dvorsky-Sahi [**9**], Kobayashi–Ørsted [**55**] and Möllers-Schwarz [**68**] to cover all cases (see also Goncharov [**24**] for the underlying Lie algebra representation and Hilgert-Kobayashi-Möllers [**38**] for a uniform construction including the full action of the Lie algebra).

The second class of groups where L^2-models of minimal representations are known are simple real Lie groups G having a Heisenberg parabolic subgroup $P = MAN$, i.e., the nilradical N is a Heisenberg group. For some of these groups G, L^2-models of minimal representation can be found in the literature, but the constructions differ from case to case, and some groups are not treated at all although they do possess a minimal representation. The probably most famous representation among them is Torasso's representation of $G = \widetilde{\mathrm{SL}}(3, \mathbb{R})$, the double cover of $\mathrm{SL}(3, \mathbb{R})$, see [**78**]. Strictly speaking, this representation is not minimal, because there is no notion of minimality for type A groups, but it is thought to correspond to the minimal nilpotent coadjoint orbit in a certain sense. Torasso's representation is realized on $L^2(\mathbb{R}^\times \times \mathbb{R})$ and he provides explicit formulas for the action of both the group G and its Lie algebra as well as for the lowest K-type. Similar formulas for the group and Lie algebra action can be found in the construction of Kazhdan and Savin [**49**] for the split simply-laced groups $\mathrm{SO}(n, n)$, $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$, although the method seems to be quite different. They construct in a natural way a representation of P on $L^2(\mathbb{R}^\times \times \Lambda)$ for some Lagrangian subspace $\Lambda \subseteq V$ of the symplectic vector space V defining the Heisenberg group N, and prove that this representation extends in a unique way to a minimal representation of G. What remains mysterious in their construction is why the extension from P to G exists. Later, Savin [**73**] applied the same construction to the group $G = G_2(\mathbb{C})$. We remark that in the case of $G = G_{2(2)}$, formulas for the Lie algebra action already appear in the work of Gelfand [**21**], but without reference to the unitary structure of this representation. Later, Sabourin [**72**] obtained a similar model for the minimal representation of $\widetilde{\mathrm{SO}}(4, 3)$ using a variant of the orbit method. Note that some of these constructions also work over non-Archimedean local fields.

While the above constructions for groups with Heisenberg parabolic subgroup are different in nature, the obtained realizations seem to be closely related. The main motivation for this work was to find a uniform construction of all these minimal representations in the spirit of the construction in the case of Siegel parabolic subgroups. The key ideas here were to exhibit the minimal representation inside a degenerate principal series $\mathrm{Ind}_P^G(\chi)$ as the kernel of a system of differential operators, and then take an appropriate Fourier transform. Roughly speaking, these ideas can indeed be applied in the Heisenberg parabolic case. In fact, several authors already noted that the minimal representation is a subrepresentation of a degenerate principal series of the form $\mathrm{Ind}_P^G(\chi)$, see [**29**, Corollary 13.7 and Proposition 14.11] for $\mathrm{SO}(4, 4)$, $E_{6(2)}$, $E_{7(-5)}$, $E_{8(-24)}$ and $G_{2(2)}$ and [**72**, Theorem 4.2.2] for $\widetilde{\mathrm{SO}}(4, 3)$ (see also [**49**, Theorem 4] for the p-adic groups D_4, E_6, E_7 and E_8). In some of these works, it is even indicated that the minimal representation is the kernel of a system of second order differential operators, but this is not pursued further.

More recently, Barchini, Kable and Zierau [**2**] constructed in a systematic way *conformally invariant systems of differential operators* on the opposite nilradical $\overline{N}$ of a Heisenberg parabolic subgroup P. As we explain in detail below, these systems are related to certain polynomial maps on the symplectic vector space defining the

Heisenberg group $\overline{N}$ that we call *symplectic invariants*. There are symplectic invariants of order one (the symplectic form), two (a moment map), three and four, and each of them gives rise to a system of differential operators of the same order whose joint kernel is a subrepresentation of a certain degenerate principal series $\mathrm{Ind}_P^G(\chi)$. In this work we focus on the system of order two and show that its joint kernel is in many cases a minimal representation. For this, we make extensive use of the structure theory for Heisenberg parabolic subalgebras developed by Slupinski-Stanton [**76, 74, 75**].

Let us remark that the joint kernel of the second order conformally invariant operators has been studied in a few examples, mostly algebraically (see e.g., the work of Kable [**44, 47, 46**] and Kubo–Ørsted [**59**]). In particular, an analysis of the corresponding representations was missing so far, probably due to the fact that the differential operators do no longer have constant coefficients but are left-invariant operators on the Heisenberg group. This suggests to consider the Heisenberg group Fourier transform in order to understand the joint kernel of these differential operators, and in this work we attempt to carry out this analysis in a uniform way in the case where the group G is non-Hermitian. For the Hermitian case we refer to a subsequent paper [**14**].

The Heisenberg group Fourier transform is more difficult to deal with since the Fourier transform of a function is operator-valued, thus adding a non-commutative flavor to the theory. Moreover, it is not clear how to define the Fourier transform of a general tempered distribution on the Heisenberg group. It turns out that for the non-compact picture of the degenerate principal series $\mathrm{Ind}_P^G(\chi)$ one can make sense of the Heisenberg group Fourier transform in the distribution sense if the character is sufficiently positive. The first key observation is that the second order conformally invariant system of Barchini-Kable-Zierau can be expressed in terms of the metaplectic representation on the Fourier transformed side (see Theorem A). This allows us to solve the corresponding system of equations on the Fourier transformed side in the distribution sense (see Theorem B) and obtain a new realization of their joint kernel (see Theorem C). Already at this point, we recognize the same formulas for the Lie algebra representation as in the work of Gelfand [**21**], Torasso [**78**] and Kazhdan-Savin [**49**], thus obtaining both a new conceptual explanation for their constructions as well as a uniform treatment. However, it is not immediate at this point whether there are K-finite vectors among the solutions, so we find explicit formulas for the functions constituting the lowest K-type in this new realization (see Theorem D). The lowest K-type generates an irreducible $(\mathfrak{g}, K)$-module which we can integrate to a minimal representation of (a finite cover of) G, realized on an L^2-space (see Theorem E). Finally, in the spirit of the work of Kazhdan-Savin [**49**] and Kobayashi-Mano [**53**], we obtain the group action of a Weyl-group element in the L^2-model which, together with a parabolic subgroup whose action is also explicit, generates G (see Theorem F).

The groups for which our construction provides L^2-models, can be divided into several classes. For the split groups $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ and $\mathrm{SO}(n,n)$, our construction yields the same model as obtained by Kazhdan-Savin [**49**] by different methods. In these cases, the minimal representation is spherical, and the expression we find for the spherical vector matches the one in **[48]** found using case-by-case computations. For $G_{2(2)}$, the model can be found in the work of Gelfand **[21]** and Savin **[73]**. Here the lowest K-type is three-dimensional. For $\mathrm{SL}(n,\mathbb{R})$ our construction actually yields two one-parameter families of representations which turn out to be unitary principal series representations induced from a different maximal parabolic subgroup. Additionally, for $n=3$ we also construct Torasso's representation which lives on the double cover $\widetilde{\mathrm{SL}}(3,\mathbb{R})$. The groups for which the obtained L^2-models seem to be new, are the quaternionic groups $E_{6(2)}$, $E_{7(-5)}$ and $E_{8(-24)}$, for which the minimal representation is an analytic continuation of the quaternionic discrete series, and the indefinite orthogonal groups $\mathrm{SO}(p,q)$ with $p\neq q$. For the latter groups, one has to assume either that $p+q$ is even or that $\min(p,q)=3$, in which case the representation actually lives on a double cover $\widetilde{\mathrm{SO}}(p,q)$. For the case $\widetilde{\mathrm{SO}}(4,3)$ our formulas do not quite match the ones by Sabourin **[72]**, but seem to be closely related (see Section 5.8.4).

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