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DYNAMICS OF QUADRATIC POLYNOMIALS, III PARAPUZZLE AND SBR MEASURES

by

Mikhail Lyubich

Dedicated to 60th birthday of A. Douady

Abstract. — This is a continuation of notes on the dynamics of quadratic polynomials. In this part we transfer our previous geometric result [L3] to the parameter plane. To any parameter value c (outside the main cardioid and the little Mandelbrot sets attached to it) we associate a "principal nest of parapuzzle pieces". We then prove that the moduli of the annuli between two consecutive pieces grow at least linearly. This implies, using Martens & Nowicki (*cf.* this volume) geometric criterion for existence of an absolutely continuous invariant measure together with [L2], that Lebesgue almost every real quadratic polynomial is either hyperbolic, or has a finite absolutely continuous invariant measure, or is infinitely renormalizable. In the further papers [L5,L7] we show that the latter set has zero Lebesgue measure, which completes the measure-theoretic picture of the dynamics in the real quadratic family.

You first plow in the dynamical plane
and then harvest in the parameter plane.
Adrien Douady

1. Introduction

This is a continuation of notes on dynamics of quadratic polynomials. In this part we transfer the geometric result of **[L3]** to the parameter plane. To any parameter value $c \in M$ in the Mandelbrot set (which lies outside of the main cardioid and satellite Mandelbrot sets attached to it) we associate a "principal nest of parapuzzle pieces"

$$\Delta^0(c) \supset \Delta^1(c) \supset \cdots$$

corresponding to the generalized renormalization type of c. Then we prove:

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Theorem A. — *The moduli of the parameter annuli* $\mathrm{mod}(\Delta^l(c) \setminus \Delta^{l+1}(c))$ *grow at least linearly.*

(See §4 for a more precise formulation.)

This result was announced at the Colloquium in honor of Adrien Douady (July 1995), and in the survey **[L4]**, Theorem 4.8. The main motivation for this work was to prove the following:

***Theorem B* (joint with Martens and Nowicki)**. — *Lebesgue almost every real quadratic* $P_c : z \mapsto z^2 + c$ *which is non-hyperbolic and at most finitely renormalizable has a finite absolutely continuous invariant measure.*

More specifically, Martens and Nowicki **[MN]** have given a geometric criterion for existence of a finite absolutely continuous invariant measure (acim) in terms of the "scaling factors". Together with the result of **[L2]** on the exponential decay of the scaling factors in the quasi-quadratic case this yields existence of the acim once "the principal nest is eventually free from the central cascades". Theorem A above implies that this condition is satisfied for almost all real quadratics which are non-hyperbolic and at most finitely renormalizable (see Theorem 5.1). Note that Theorem A also implies that this condition is satisfied on a set of positive measure, which yields a new proof of Jacobson's Theorem **[J]** (see also Benedicks & Carleson **[BC]**).

A measure μ will be called SBR (Sinai-Bowen-Ruelle) if

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x} \to \mu \tag{1.1}$$

for a set of x of positive *Lebesgue* measure. It is known that if an SBR measure exists for a real quadratic map $f = P_c$, $c \in [-2, 1/4]$, on its invariant interval I_c, then it is unique and (1.1) is satisfied for Lebesgue almost all $x \in I_c$ (see Introduction of **[MN]** for a more detailed discussion and references). Theorem B yields

Corollary. — *For almost all* $c \in [-2, 1/4]$, *the quadratic polynomial* P_c *has a unique SBR measure on its invariant interval* I_c.

Another consequence of our geometric results is concerned with the shapes of little Mandelbrot copies (see **[L3]**, §2.5, for a discussion of little Mandelbrot copies). Let us say that a Mandelbrot set M' has a (K, ε)-*a bounded shape* if the straightening $\chi : M' \to M$ admits a K-quasi-conformal extension to an $(\varepsilon \operatorname{diam} M')$-neighborhood of M'. We say that the little Mandelbrot sets of some family *have bounded shape* if a bound (K, ε) can be selected uniform over the family.

A Mandelbrot copy M' is called *maximal* if it is not contained in any other copy except M itself. It is called *real* if it is centered at the real line.

A little Mandelbrot copy encodes the combinatorial type of the corresponding renormalization. In **[L3]** we dealt with diverse numerical functions of the combinatorial type. For real copies a crucial information is encoded by the *essential period* $p_e(M')$ (see **[L3**, §8.1], **[LY]**).

For a definition of Misiurewicz wakes see §3.3 of this paper.

Theorem C. — *For any Misiurewicz wake O, the maximal Mandelbrot copies contained in O have bounded shape. In particular, all maximal real Mandelbrot copies, except the doubling one, have a bounded shape. Moreover, a real copy M' has a (K, ε)-bounded shape, where $K \to 1$ and $\varepsilon \to \infty$ as $p_e(M') \to \infty$.*

In §6 we will refine this statement and will comment on its connection with the MLC problem and the renormalization theory.

Let us now take a closer look at Theorem A. It nicely fits to the general philosophy of correspondence between the dynamical and parameter plane. This philosophy was introduced to holomorphic dynamics by Douady and Hubbard **[DH1]**. Since then, there have been many beautiful results in this spirit, see Tan Lei **[TL]**, Rees **[R]**, Shishikura **[Sh]**, Branner-Hubbard **[BH]**, Yoccoz (see **[H]**).

In the last work, special tilings into "parapuzzle pieces" of the parameter plane are introduced. Its main geometric result is that the tiles around at most finitely renormalizable points shrink. It was done by transferring, in an ingenious way, the corresponding dynamical information into the parameter plane.

In **[L3]** we studied the rate at which the dynamical tiles shrink. The main geometric result of that paper is the linear growth of the moduli of the principal dynamical annuli. Let us note that the way we transfer this result to the parameter plane (Theorem A) is substantially different from that of Yoccoz. Our main conceptual tool is provided by holomorphic motions whose transversal quasi-conformality is responsible for commensurability between the dynamical and parameter pictures (compare Shishikura **[Sh]**). To make it work we exploit existence of uniform quasi-conformal pseudo-conjugacy between the generalized renormalizations **[L3]**.

The properties of holomorphic motions are discussed in §2. In §3 we describe the principal parameter tilings according to the generalized renormalization types of the maps. In §4 we prove Theorem A. In §5, we derive the consequence for the real quadratic family (Theorem B). In the last section, §6, we prove Theorem C on the shapes of Mandelbrot copies.

Let us finally draw the reader's attention to the work of LeRoy Wenstrom **[W]** which studies in detail parapuzzle geometry near the Fibonacci parameter value.

Remark. — We have recently proven that the set of infinitely renormalizable real parameter values has zero linear measure. Together with Theorem B this implies that almost every real quadratic has either an attracting cycle or an absolutely continuous invariant measure **[L7]**.

2. Background

2.1. Notations and terminology

$$\begin{aligned}
\mathbb{D}_r(p) &= \{z : |z-p| < r\};\\
\mathbb{D}_r &\equiv \mathbb{D}_r(0);\\
\mathbb{D} &\equiv \mathbb{D}_1;\\
\mathbb{T}_r &= \{z : |z| = r\};\\
\mathbb{A}(r,R) &= \{r < |z| < R\}.
\end{aligned}$$

The closed and semi-closed annuli are denoted accordingly: $\mathbb{A}[r,R]$, $\mathbb{A}(r,R]$, $\mathbb{A}[r,R)$.

By a *topological disc* we will mean a simply connected domain $D \subset \mathbb{C}$ whose boundary is a Jordan curve.

Let π_1 and π_2 denote the coordinate projections $\mathbb{C}^2 \to \mathbb{C}$. Given a set $\mathbb{X} \subset \mathbb{C}^2$, we denote by $X_\lambda = \pi_1^{-1}\{\lambda\}$ its vertical cross-section through λ (the "fiber" over λ). Vice versa, given a family of sets $X_\lambda \subset \mathbb{C}$, $\lambda \in D$, we will use the notation:

$$\mathbb{X} = \cup_{\lambda\in D} X_\lambda = \{(\lambda, z) \in \mathbb{C}^2 : \lambda \in D, z \in X_\lambda\}.$$

Let us have a discs fibration $\pi_1 : \mathbb{U} \to D$ over a topological disc $D \subset \mathbb{C}$ (such that the sections U_λ are topological discs, and the closure of $\mathbb{U}$ in $D \times \mathbb{C}$ is homeomorphic to $D \times \overline{\mathbb{D}}$ over D). In this situation we call $\mathbb{U}$ an (open) *topological bidisc* over D. We say that this fibration admits an extension to the boundary ∂D if the closure $\overline{\mathbb{U}}$ of $\mathbb{U}$ in $\mathbb{C}^2$ is homeomorphic over $\overline{D}$ to $\overline{D} \times \overline{\mathbb{D}}$. The set $\overline{\mathbb{U}}$ is called a (closed) bidisc. We keep the notation $\mathbb{U}$ for the fibration of *open* discs over the closed disc $\overline{D}$ (it will be clear from the context over which set the fibration is considered).

If $U_\lambda \ni 0$, $\lambda \in D$, we denote by $\mathbf{0}$ the zero section of the fibration.

Given a domain $\Delta \subset D$, let $\mathbb{U}|\Delta = \mathbb{U} \cap \pi_1^{-1}\Delta$. This is a bidisc over Δ.

If the fibration π_1 admits an extension over the boundary ∂D, we define the *frame* $\delta\mathbb{U}$ as the topological torus $\cup_{\lambda\in\partial D}\partial U_\lambda$. A section $\mathbf{\Phi} : D \to \mathbb{U}$ is called *proper* if it is continuous up to the boundary and $\mathbf{\Phi}(\partial D) \subset \delta U$.

We assume that the reader is familiar with the theory of quasi-conformal maps (see e.g., [**A**]). We will use a common abbreviation K-qc for "K-quasi-conformal". Dilatation of a qc map h will be denoted as $\mathrm{Dil}(h)$.

Notation $a_n \asymp b_n$ means, as usual, that the ratio a_n/b_n is positive and bounded away from 0 and ∞.

2.2. Holomorphic motions. — Given a domain $D \subset \mathbb{C}$ with a base point $*$ and a set $X_* \subset \mathbb{C}$, a *holomorphic motion* $\boldsymbol{h}$ of X_* over D is a family of injections $h_\lambda : X_* \to \mathbb{C}$, $\lambda \in D$, such that $h_* = \mathrm{id}$ and $h_\lambda(z)$ is holomorphic in λ for any $z \in X_*$. We denote $X_\lambda = h_\lambda X_*$. The restriction of $\boldsymbol{h}$ to a parameter domain $\Delta \subset D$ will be denoted as $\boldsymbol{h}|\Delta$.