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SKEW DIFFERENTIAL FIELDS, DIFFERENTIAL AND DIFFERENCE EQUATIONS

by

Marius van der Put

Dedicated to Jean-Pierre Ramis on the occasion of his 60th birthday

Abstract. — The central question is: Let a differential or difference equation over a field K be isomorphic to all its Galois twists w.r.t. the group $\mathrm{Gal}(K/k)$. Does the equation descend to k? For a number of categories of equations an answer is given.

***Résumé* (Corps différentiels non commutatifs, équations différentielles et équations aux différences)**

On étudie la descente sur un corps k d'une équation différentielle ou aux différences donnée sur un corps K et qui est isomorphe à toutes ses conjuguées sous l'action du groupe de Galois $\mathrm{Gal}(K/k)$ de K sur k. On traite le cas de plusieurs classes d'équations.

Introduction

Rationality questions for differential modules and differential operators are strongly related to skew differential fields. This theme has been developed in [**H-P**]. An open question in [**H-P**] has found an answer, namely the existence and unicity of the differentiation on a skew field of finite dimension over its center, that is, a differential field in the usual sense. The present paper, *written in honour of Jean-Pierre Ramis*, reviews these descent problems but now in the context of meromorphic differential equations. A remarkable family of examples is the result. Equally surprising is that descent does hold for meromorphic q-difference equations. This is shown using recent work of J.-P. Ramis and J. Sauloy on moduli for these equations. Finally, it is shown that descent does not hold for meromorphic ordinary difference equations.

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1. The construction of skew differential fields

Let k denote a differential field having characteristic 0. The skew fields, or more generally the algebras F, that we consider here are central, simple, finite dimensional over their center k. A differentiation on F, extending the differentiation of k, is an additive map $\partial : F \to F$ such that $\partial(ab) = \partial(a)b + a\partial(b)$ holds for all $a, b \in F$. Moreover, we require that $\partial(a) = a'$ for every $a \in k$. For special cases, such differentiations are constructed in [**H-P**]. Here we prove a general result on differentiations.

Theorem 1.1. — *Let k be a differential field of characteristic 0. Let F be a central, simple algebra over k of finite dimension. Suppose that F contains a maximal commutative subfield K which is Galois over k. There exists a differentiation ∂_0 on F extending the differentiation of k and having the property $\partial_0 K \subset K$.*

Moreover, for any differentiation ∂ on F extending the differentiation of k, there is an element $c \in F$ (unique up to an element in k) such that $\partial(a) = \partial_0(a) + ac - ca$ for all $a \in F$.

Proof. — The asumptions on F imply that F is a crossed product algebra (see [**Bl**], Chapitre IV). The structure of F is the following:

The elements of F are uniquely given by expressions $\sum_{\sigma \in G} d_\sigma[\sigma]$, where G is the Galois group of K/k and all $d_\sigma \in K$. The multiplication is given by the rules $[\sigma]d = \sigma(d)[\sigma]$ (for $\sigma \in G$, $d \in K$) and $[\sigma][\tau] = c(\sigma, \tau)[\sigma\tau]$. Here $(\sigma, \tau) \mapsto c(\sigma, \tau)$ is a 2-cocycle representing an element of $H^2(G, K^*)$.

Let $'$ denote the unique differentiation on K, extending the one of k. Then $(\sigma, \tau) \mapsto c(\sigma, \tau)'/c(\sigma, \tau)$ is a 2-cocycle for $H^2(G, K)$. Since the latter group is trivial (see [**Se**]), there are elements $\{a(\sigma)\}$ in K such that $a(\sigma) + {}^{\sigma}a(\tau) - a(\sigma\tau) = c(\sigma, \tau)'/c(\sigma, \tau)$. Now, define ∂_0 by the formula

$$\partial_0\big(\sum d_\sigma[\sigma]\big) = \sum(d'_\sigma + d_\sigma a(\sigma))[\sigma].$$

The verification that ∂_0 has the required properties is straightforward.

Let ∂ be another derivation on F extending the one of k. Then $\partial - \partial_0$ is a k-linear derivation on F. It is known that these derivations are given by $a \mapsto [a, c] := ac - ca$ for $c \in F$. (See [**Ren**], Corollaire 3 on p. 111). □

We note that the differentiation ∂ on F, extending the differentiation of k, is *almost unique* if one prescribes that ∂ is the usual differentiation on the maximal commutative subfield K of F. Indeed, $\partial(a) = \partial_0(a) + [a, c]$ for some $c \in F$. For $a \in K$ one has $[a, c] = 0$. Further, K is a maximal commutative subfield of F and, thus, $c \in K$.

2. Skew differential fields over $\mathbf{R}(\{x\})$

Notations. — $k := \mathbf{R}(\{x\})$, $K := \mathbf{C}(\{x\})$ are the fields of convergent Laurent series over $\mathbf{R}$ and $\mathbf{C}$. The differentiation on these fields is given by $f \mapsto f' := x df/dx$.

Hamilton's quaternion field is denoted by $\mathbf{H} := \mathbf{R}1 + \mathbf{R}i + \mathbf{R}j + \mathbf{R}k$. Then $F = \mathbf{H} \otimes_{\mathbf{R}} k$ is a quaternion field with center k. Let $\| \ \|$ denote the usual norm on $\mathbf{H}$. The differentiation on F is defined by $(h \otimes f)' = h \otimes f'$ for all $h \in \mathbf{H}$ and $f \in k$. The elements of F are represented by convergent Laurent series with coefficients in $\mathbf{H}$. Thus, an element of F has the form $\sum a_n x^n$ with all $a_n \in \mathbf{H}$ and such that only finitely many negative powers of x are present and, moreover, there are positive constants C, R with $\|a_n\| \leqslant CR^n$ for all $n \geqslant 0$. One observes that $(\sum a_n x^n)' = \sum n a_n x^n$.

Consider the 1-dimensional differential module $M = Fe$ over F, defined by the formula $\partial e = de$. After identifying F with M, via $v \mapsto ve$, one has $\partial(v) = v' + vd$. For d we make the choice $d = i + x^{-1}j$. One can consider M as a differential module over K of dimension 2, by the obvious inclusion $K \subset F$. Further, M is also a differential module over k of dimension 4.

Proposition 2.1. — $\mathrm{End}_{k[\partial]}(M)$, *the* $\mathbf{R}$*-algebra of the endomorphisms of the* k*-differential module* M, *is equal to* $\mathbf{H}$.

Proof. — Every k-linear map $L : M \to M$ has uniquely the form $L(v) = va_0 + iva_1 + jva_2 + kva_3$ with $a_0, \ldots, a_3 \in F$. A calculation shows that

$$(\partial L - L\partial)(v) = v(a_0' + [a_0, d]) + iv(a_1' + [a_1, d]) + jv(a_2' + [a_2, d]) + kv(a_3' + [a_3, d]).$$

Hence $L \in \mathrm{End}_{k[\partial]}(M)$ if and only if $a_i' + [a_i, d] = 0$ for $i = 0, \ldots, 3$. Therefore, the proposition follows from the statement:

The only solutions $a \in F$ *of* $a' + [a, d] = 0$ *are* $a \in \mathbf{R}$.

The proof of this statement is as follows. Write $a = \sum a_n x^n$ with all $a_n \in \mathbf{H}$. Then, $a' + [a, d] = 0$ translates into

$$\sum (na_n + [a_n, i] + [a_{n+1}, j])x^n = 0.$$

For $n > 0$ and $t = t_0 + t_1 i + t_2 j + t_3 k \in \mathbf{H}$ one has

$$nt + [t, i] = nt_0 + nt_1 i + (nt_2 + 2t_3)j + (nt_3 - 2t_2)k.$$

It follows that $\|nt + [t, i]\| \geqslant n\|t\|$.

For $s = s_0 + s_1 i + s_2 j + s_3 k \in \mathbf{H}$ one has $[s, j] = 2s_i k - 2s_3 i$ and thus $\|[s, j]\| \leqslant 2\|s\|$. One concludes that for $n > 0$ one has $\|a_{n+1}\| \geqslant \frac{n}{2}\|a_n\|$. If some $a_m \neq 0$ with $m > 0$, then $a_n \neq 0$ for all $n \geqslant m$. Moreover, for a suitable constant $C > 0$ one has $\|a_n\| \geqslant C2^{-n}n!$ for all $n \geqslant m$. This contradicts the assumption that the Laurent series a is convergent. The conclusion is that $a_n = 0$ for all $n \geqslant 1$.

$0 \cdot a_0 + [a_0, i] + [a_1, j] = 0$ implies that $a_0 \in \mathbf{R} + \mathbf{R}i$. After subtracting from a a real number, one has $a_0 \in \mathbf{R}i$. In the sequel we will write $*$ for a non-zero real number. Suppose that $a_0 = *i$. Then $-a_{-1} + [a_{-1}, i] + [a_0, j] = 0$ implies that $a_{-1} = *j + *k$. The equation $-2a_{-2} + [a_{-2}, i] + [a_{-1}, j] = 0$ implies $a_{-2} = *i$. By induction, one finds that $a_{-2m} = *i$ and $a_{-2m-1} = *j + *k$. This contradicts the fact that a is a Laurent series. One concludes that $a_0 = 0$. $\square$

In the proof of the following corollary we will use some ideas and results of [**H-P**]. For any differential fields $k \subset K$, one says that a differential module M over K descends to k, if there exists a differential module N over k such that $M \cong K \otimes_k N$. Suppose that K/k is a Galois extension with group G. For a differential module M over K and for $\sigma \in G$, one defines the twisted differential module ${}^\sigma M$ by:

- ${}^\sigma M$ is equal to M as an additive group.
- For $f \in K$ and $m \in {}^\sigma M$ one puts $fm = \sigma^{-1}(f)m$.
- The ∂ on ${}^\sigma M$ coincides with ∂ on M.

If M descends to k, then clearly ${}^\sigma M \cong M$ for all $\sigma \in G$. The descent problem of [**H-P**] asks whether the converse is true. In general, there is an obstruction given by the class of a 2-cocycle.

Corollary 2.2. — *We keep the above notations.*

(a) *$M = Fe$ is an irreducible differential module over k.*

(b) *Let σ denote the non-trivial element of the Galois group of K/k. Then the twisted differential module ${}^\sigma M$ over K is isomorphic to M.*

(c) *The K-differential module M does not descend to k.*

(d) *Let $\widehat{k} = \mathbf{R}((x))$ and $\widehat{K} = \mathbf{C}((x))$. The $\widehat{K}$-differential module $\widehat{K} \otimes_K M$ descends to $\widehat{k}$.*

Proof

(a) Suppose that M is reducible as a K-differential module. Let $N \subset M$ be a 1-dimensional K-submodule. Then jN is also a 1-dimensional K-submodule and $M = N + jN$. In particular, M is semi-simple as K-differential module. If M is irreducible as K-differential module, then M is semi-simple, too. According to [**H-P**], proposition 2.7, M is also semi-simple as k-differential module. Since $\mathrm{End}_{k[\partial]}(M)$ is a skew field, one has that M is irreducible as k-differential module.

(b) The map $\Phi(\sigma) : M \to M$, given by $fe \mapsto jfe$, is a σ-linear bijection commuting with ∂. This proves the statement.

(c) Since $\Phi(\sigma)\Phi(\sigma) = -1$, the 2-cocycle class in $H^2(\{1, \sigma\}, \mathbf{C}^*)$, associated to M, is not trivial. It follows from [**H-P**], theorem 2.8, that the K-differential module M does not descend to k.

(d) Put $\widehat{M} = \widehat{K} \otimes_K M$. The twisted $\widehat{K}$-module ${}^\sigma \widehat{M}$ is isomorphic to $\widehat{M}$. According to [**H-P**], theorem 2.4, $\widehat{M}$ descends to $\widehat{k}$. □

Explicit calculations. — The element e of the K-differential module $M = Fe$ is a cyclic vector. The minimal monic operator $L_2 \in K[\partial]$ with $L_2 e = 0$ can be calculated to be

$$\delta^2 + \delta + (1 + x^{-2} - i).$$

Here, we prefer to write $\delta = xd/dx$ instead of ∂, since the latter may be confusing. Note that $\delta x = x\delta + x$. The corollary translates into the following: