Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 152
Fascicule 1

2024

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 19-31

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 1, tome 152, mars 2024

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 160 € ($ 240),
avec supplément papier : Europe 244 €, hors Europe 330 € ($ 421)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
152 (1), 2024, p. 19-31

TRANSIENCE IN LAW FOR SYMMETRIC RANDOM WALKS IN INFINITE MEASURE

by Timothée Bénard

Abstract. — We consider a random walk on a second countable locally compact topological space endowed with an invariant Radon measure. We show that if the walk is symmetric and if every subset that is invariant by the walk has zero or infinite measure, then one has escape of mass for almost every starting point. We then apply this result in the context of homogeneous random walks on infinite volume spaces and deduce a converse to the Eskin–Margulis recurrence theorem.

Résumé (*Transience en loi des marches aléatoires symétriques en mesure infinie*). — On considère une marche aléatoire sur un espace topologique localement compact à base dénombrable muni d'une mesure de Radon invariante. On montre que si la marche est symétrique et si tout sous-ensemble invariant par la marche est de mesure nulle ou infinie, alors il y a fuite de masse pour presque tout point de départ. Nous appliquons ensuite ce résultat dans le contexte des marches aléatoires homogènes en volume infini, et déduisons une réciproque au théorème de récurrence d'Eskin-Margulis.

1. Introduction

The starting point of this text is an article published by Eskin and Margulis in 2004, which studies the recurrence properties of random walks on homogeneous spaces [11]. The space in question is a quotient G/Λ, where G is a real Lie group and $\Lambda \subseteq G$ a discrete subgroup. Given a probability measure

Texte reçu le 7 janvier 2022, modifié le 24 mai 2022, accepté le 28 septembre 2022.

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Mathematical subject classification (2020). — 37A50, 37B20.

Key words and phrases. — Transience, random walks, Lie groups.

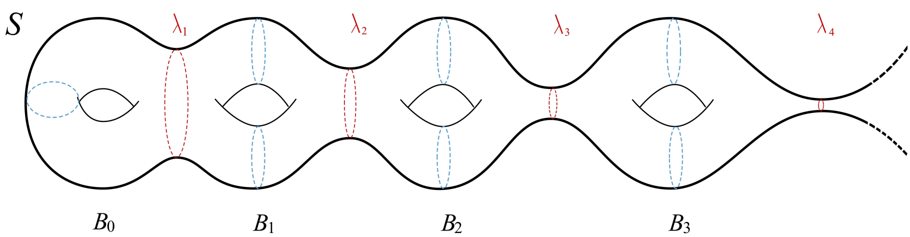
doi:10.24033/bsmf.2883

μ on G, we can define a random walk on G/Λ with transitional probability measures $(\mu * \delta_x)_{x\in G/\Lambda}$. In more concrete terms, a random step starting at a point $x \in G/\Lambda$ is performed by choosing an element $g \in G$ randomly with law μ and letting it act on x. The two authors ask about the position of the walk at time n for large values of n. They manage to show a surprising result: if G is a simple real algebraic group, if Λ has finite covolume in G, and if the support of μ is compact and generates a Zariski-dense subgroup of G, then for every starting point $x \in G/\Lambda$, the n-th step distribution of the walk $\mu^{*n} * \delta_x$ does not escape at infinity. More precisely, all the weak-$*$ limits of $(\mu^{*n} * \delta_x)_{n\geq 0}$ have mass 1. One says that there is no escape of mass. This reminds us of the behavior of the unipotent flow as highlighted by Dani and Margulis in [8, 13], proving that the trajectories of a unipotent flow on G/Λ spend most of their time inside compact sets. Eskin–Margulis' result is actually the starting point of a fruitful analogy with Ratner theorems, which led to the classification of stationary probability measures on X thanks to the work of Benoist and Quint [3, 4], followed by Eskin and Lindenstrauss [10].

This paper asks the question of a converse to Eskin–Margulis theorem.

> *Is the absence of mass escape characteristic of random walks on homogeneous spaces of finite volume, or could it also happen for walks in infinite volume?*

Let us illustrate the question with an *example*. Consider S a hyperbolic surface of the form



Such a surface is made of blocks $(B_i)_{i\geq 0}$ glued together along geodesic circles (in red) of respective length $(\lambda_i)_{i\geq 1} \in \mathbb{R}_{>0}^{\mathbb{N}^*}$. Each block comes with a pants decomposition, whose internal boundary components (in blue) are assumed to have length 1. We consider a (discretized) Brownian motion on (the unit tangent bundle of) S starting from B_0. If all the λ_i's are equal, the walk looks like the nearest neighbor random walk on $\mathbb{N}$, so we expect escape of mass. On the other hand, in the degenerate case where some λ_i is equal to 0, the walk evolves in a finite volume space so there is no mass escape by the Eskin–Margulis theorem, or more simply ergodic considerations in this case. Now, we may wonder what happens in intermediate situations where the sequence $(\lambda_i)_{i\geq 1}$ is positive but allowed to go to zero extremely fast. We will see that the n-th step distribution of the walk always escapes at infinity, regardless of the choice of $(\lambda_i)_{i\geq 1} \in \mathbb{R}_{>0}^{\mathbb{N}^*}$ (Theorem 1.2).

In Section 2, we establish escape of mass in a very general framework, which does not rely on the algebraic setting mentioned previously. The measure μ is assumed to be symmetric, i.e., invariant under the inversion map $g \mapsto g^{-1}$.

Theorem 1.1. — *Let X be a locally compact second countable topological space equipped with a Radon measure λ, let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ, and let μ be a probability measure on Γ whose support generates Γ as a closed group.*

If the probability measure μ is symmetric and if every measurable Γ-invariant subset of X has zero or infinite λ-measure, then for λ-almost every starting point $x \in X$, one has the weak-$*$ convergence:

$$\mu^{*n} * \delta_x \xrightarrow[n\to+\infty]{} 0.$$

To put it in a nutshell, a symmetric random walk on a measured space without finite volume invariant subset is transient in law for almost every starting point. This result can be seen as an analogue in infinite measure of equidistribution results for random walks in finite measure obtained independently by Rota [16] and Oseledets [15].

In our statement, a measurable subset $A \subseteq X$ is considered as Γ-invariant if for every $g \in \Gamma$, $\lambda(gA\Delta A) = 0$. We will see later an equivalent characterization in terms of the Markov operator of the walk (Lemma 2.4).

Note also that the condition of symmetry on μ plays a role. Let (X, λ) be a locally compact space with an infinite Radon measure and endowed with a conservative ergodic measure-preserving $\mathbb{Z}$-action. If $\mu = \delta_1$ is the Dirac mass at $1 \in \mathbb{Z}$, then for λ-almost every $x \in X$, the sequence $(n.x)_{n\geq 0}$ comes back close to x infinitely often, so $\mu^{*n} * \delta_x = \delta_{n.x}$ cannot weakly converge to 0.

Without symmetry assumption on μ, the proof of Theorem 1.1 still yields convergence to 0 in Cesàro-averages.

In Section 3, we use Theorem 1.1 to address our original question concerning the escape of mass of homogeneous walks on infinite volume spaces. We obtain the following result.

Theorem 1.2. — *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a discrete subgroup of infinite covolume in G, and μ a probability measure on G whose support generates a group with unbounded projections in the noncompact factors of G.*

Then for almost every $x \in G/\Lambda$, one has the weak-$*$ convergence:

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x \xrightarrow[n\to+\infty]{} 0. \tag{1}$$

Moreover, if the probability measure μ is symmetric, then the convergence can be strengthened:

$$\mu^{*n} * \delta_x \xrightarrow[n\to+\infty]{} 0. \tag{2}$$

Note that convergence (1) is sufficient to ensure that Eskin–Margulis' observations cannot occur when the quotient G/Λ has infinite measure. Indeed, for almost every $x \in G/\Lambda$, we obtain the existence of an extraction $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\mu^{*\sigma(n)} * \delta_x \xrightarrow[n\to+\infty]{} 0.$$

Theorem 1.2 describes the asymptotic behavior of the probabilities of position for *almost every* starting point $x \in G/\Lambda$. One may not hope for transience in law for every starting point as it is possible that the orbit $\Gamma.x$ is finite.

To conclude this introduction, we emphasize that our paper focuses on the behavior *in law* of a random walk on G/Λ. A related natural theme of study is the behavior of the walk trajectories for which analogous notions of recurrence or transience exist. Although our conclusions support the idea that walks in infinite volume are always transient in law (the mass escapes), the picture becomes mixed when it comes to considering walk trajectories. Indeed, as observed in [7] or [2], pointwise recurrence or transience also depends on the nature of the ambient space.

2. A general result of transience in law

This section is dedicated to the proof of Theorem 1.1. The proof results from a combination of the Dunford–Schwartz theorem [9] and Akcoglu–Sucheston's pointwise convergence of alternating sequences [1]. The latter guarantees that for λ-almost every $x \in X$, the sequence of probability measures $(\mu^{*n} * \check{\mu}^{*n} * \delta_x)_{n\geq 0}$ weak-$*$ converges toward a finite measure and is based on Rota and Oseledets' original idea to express this alternating sequence in terms of reversed martingales [16, 15]. We give a shorter proof than the one in [1]. Although our proof follows very closely the one of Rota [16] who considered walks on finite volume spaces, we use a different formalism that may be useful to illustrate the technique of "equidistribution of fibers" contained in the work of Benoist–Quint [4] (see also [5]).

2.1. Backwards martingales. — We first present a convergence theorem for backwards martingales on a σ-finite measured space. It will play a crucial role in the proof of the convergence of back-and-forths (2.2).

First, let us recall the definition of conditional expectation.

Definition (Conditional expectation). — Let $(\Omega, \mathcal{F})$ be a measurable space, $\mathcal{Q}$ a sub-σ-algebra of $\mathcal{F}$, and m a positive measure on $(\Omega, \mathcal{F})$ whose restriction

$m_{|\mathcal{Q}}$ is σ-finite. Then, for every function $f \in L^1(\Omega, \mathcal{F}, m)$, there exists a unique function $f' \in L^1(\Omega, \mathcal{Q}, m)$ such that for all $\mathcal{Q}$-measurable subset $A \in \mathcal{Q}$, one has $m(f\, 1_A) = m(f'\, 1_A)$. We denote this function by $\mathbb{E}_m(f|\mathcal{Q})$.

We have the following [12, page 533] (see also [6]).

Theorem 2.1 (Convergence of backwards martingales). — *Let $(\Omega, \mathcal{F}, m)$ be a measured space, $(\mathcal{Q}_n)_{n\geq 0}$ a decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that for all $n \geq 0$, the restriction $m_{|\mathcal{Q}_n}$ is σ-finite. Then, for any function $f \in L^1(\Omega, \mathcal{F}, m)$, there exists $\psi \in L^1(\Omega, \mathcal{F}, m)$ such that we have the almost sure convergence:*

$$\mathbb{E}_m(f|\mathcal{Q}_n) \xrightarrow[n\to+\infty]{} \psi \qquad m\text{-a.e.}$$

Remark. If the measure m is σ-finite with respect to the tail-algebra $\mathcal{Q}_\infty := \bigcap_{n\geq 0} \mathcal{Q}_n$, then Theorem 2.1 can be deduced from the probabilistic case (by restriction to $\mathcal{Q}_\infty$-measurable domains of finite measure), and we can certify that $\psi = \mathbb{E}_m(f|\mathcal{Q}_\infty)$. On the opposite extreme, if every $\mathcal{Q}_\infty$-measurable subset of Ω has m-measure 0 or $+\infty$, then, the integrability of ψ implies that $\psi = 0$. The general picture is a direct sum of these two contrasting situations as $\Omega = \Omega_\sigma \amalg \Omega_\infty$ where Ω_σ is a countable union of $\mathcal{Q}_\infty$-measurable sets of finite measure, and the restricted measure $m_{|\Omega_\infty}$ takes only the values 0 or $+\infty$ on $\mathcal{Q}_\infty$ (see [12], footnote of page 533).

2.2. Convergence of back-and-forths. — We now state and show Theorem 2.2 about the convergence of back-and-forths of the μ-random walk on X. We denote by $\check{\mu} := i_*\mu$ the image of μ under the inversion map $i : \Gamma \to \Gamma$, $g \mapsto g^{-1}$.

Theorem 2.2 (Convergence of back-and-forths [1]). — *Let X be a locally compact second countable topological space equipped with a Radon measure λ, let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ, and let μ be a probability measure on Γ.*

There exists a family $(\nu_x)_{x\in X}$ of finite measures on X such that for λ-almost every $x \in X$, one has the weak-$*$ convergence:

$$(\mu^{*n} * \check{\mu}^{*n}) * \delta_x \xrightarrow[n\to+\infty]{} \nu_x.$$

Proof. — The following proof is inspired by [16] and [4]. Denote

$$B := \Gamma^{\mathbb{N}^*}, \quad \beta := \mu^{\mathbb{N}^*}, \quad T : B \to B, (b_i)_{i\geq 1} \mapsto (b_{i+1})_{i\geq 1}$$

the one-sided shift. One introduces a σ-finite fibered dynamical system (B^X, β^X, T^X) setting

- $B^X := B \times X$
- $\beta^X := \beta \otimes \lambda \in \mathcal{M}^{Rad}(B \times X)$
- $T^X : B^X \to B^X, (b, x) \mapsto (Tb, b_1^{-1}x)$.