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Pierre-Louis Blayac

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

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THE BOUNDARY OF RANK-ONE DIVISIBLE CONVEX SETS

by Pierre-Louis Blayac

Abstract. — We prove that for any non-symmetric irreducible divisible convex set, the proximal limit set is the full projective boundary.

Résumé (*Le bord des convexes divisibles de rang un*). — Nous démontrons que pour tout convexe divisible irréductible non symétrique, l'ensemble limite proximal est le bord projectif tout entier.

1. Introduction

This note concerns the rich topic of divisible convex sets, which began more than 60 years ago with the work of Kuiper [17] and Benzécri [8] and is today very active. We refer to Benoist's survey [6], which presents many interesting results and shows how diverse the mathematics interacting with this topic are. Let us fix for the whole paper a finite-dimensional real vector space V. A subset of the projective space $\mathrm{P}(V)$ is *properly convex* if it is convex and bounded in some affine chart. A properly convex open subset $\Omega \subset \mathrm{P}(V)$ is *divisible* if it is *divided* by some discrete subgroup of $\Gamma \subset \mathrm{PGL}(V)$, i.e. Γ acts cocompactly

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Pierre-Louis Blayac, Department of Mathematics, University of Michigan, 530 Church Street Ann Arbor, MI 48103-1043, USA • *E-mail :* blayac@umich.edu

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on Ω. We denote by $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(V)$ the closed subgroup consisting of the elements g that preserve Ω.

1.1. Structural results on divisible convex sets. — Let us begin by recalling a series of structural results on divisible convex sets that relates various type of properties: analytical, algebraic and dynamical ones. More precisely, given a divisible convex set Ω, regularity properties of its boundary $\partial\Omega$, algebraic ones of $\mathrm{Aut}(\Omega)$ and its cocompact subgroups, as well as dynamical ones of the action on the projective space of $\mathrm{Aut}(\Omega)$ and its subgroups are intertwined.

One cornerstone of these structural results is the following result due to Vey [19, Th. 3]. Consider a divisible convex set $\Omega \subset \mathrm{P}(V)$. Then

- either there exists two proper subspaces $V_1, V_2 \subset V$ with $V = V_1 \oplus V_2$ and two properly convex open cones $C_1 \subset V_1$ and $C_2 \subset V_2$ such that $\mathrm{P}(C_1) \subset \mathrm{P}(V_1)$ and $\mathrm{P}(C_2) \subset \mathrm{P}(V_2)$ are divisible convex sets and $\Omega = \mathrm{P}(C_1 + C_2)$ — in this case Ω is said to be *reducible*;
- or any cocompact closed subgroup of $\mathrm{Aut}(\Omega)$ is *strongly irreducible*, in the sense that it does not preserve any finite union of proper subspaces of $\mathrm{P}(V)$ — in this case Ω is said to be *irreducible*.

Let us assume that Ω is irreducible. Combining work of Koecher [16], Vinberg [20] and Benoist [3] yields the following dichotomy:

- either $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(V)$ is a semi-simple Lie subgroup that acts transitively on Ω, in which case Ω is called *symmetric*;
- or $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(V)$ is a discrete Zariski-dense subgroup.

If Ω is symmetric, then it naturally identifies with the Riemannian symmetric space of $\mathrm{Aut}(\Omega)$, and there is yet another natural dichotomy: namely, either $\mathrm{Aut}(\Omega)$ has real rank 1, in which case Ω is an ellipsoid and $\mathrm{Aut}(\Omega)$ is isomorphic to $\mathrm{PO}(n,1)$ for $n = \dim(V) - 1$, or $\mathrm{Aut}(\Omega)$ has real rank greater than one, it is isomorphic to $\mathrm{PGL}(n, \mathbb{K})$ for some $n \geq 3$, and for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or the classical division algebra of quaternions, or of octonions if $n = 3$ (see, for instance, [6, §2.4]).

Recently, A. Zimmer proved the following higher-rank rigidity result [21, Th. 1.4], analogous to a celebrated result in Riemannian geometry by Ballmann [1] and Burns–Spatzier [12]. If Ω is not symmetric, then it is *rank-one* in the following sense.

Definition 1.1. — A divisible convex set $\Omega \subset \mathrm{P}(V)$ is said to be rank-one if there exists in $\partial\Omega$ a *strongly extremal point*, namely a point $\xi \in \partial\Omega$ such that $[\xi, \eta] \cap \Omega$ is non-empty for any $\eta \in \partial\Omega \smallsetminus \{\xi\}$ (in other words, ξ is "visible" from any other point of the projective boundary).

The notion of rank-one divisible convex sets (and more generally of rank-one geodesics, automorphisms, groups of automorphisms, quotients of properly convex open sets, which we do not define here) was developed by M. Islam [14] and Zimmer [21], who established other characterisations of this property; see also [10, 9] for more characterisations.

It is elementary to check that reducible divisible convex sets and symmetric irreducible divisible convex sets with higher-rank automorphism groups are not rank-one (see, e.g. [10, §2.7 & §7]). These convex sets are hence called *higher-rank*. On the other hand, ellipsoids are rank-one.

1.2. The proximal limit set. — Let $\Omega \subset \mathrm{P}(V)$ be an irreducible divisible convex set. The present note concerns an important $\mathrm{Aut}(\Omega)$-invariant compact subset of the projective boundary $\partial\Omega$, called the *proximal limit set* and denoted by $\Lambda_\Omega^{\mathrm{prox}}$. Recall that a projective transformation $g \in \mathrm{PGL}(V)$ is called *proximal* if it has an attracting fixed point in $\mathrm{P}(V)$.

Definition 1.2. — Let $\Omega \subset \mathrm{P}(V)$ be an irreducible divisible convex set. The proximal limit set of Ω is the closure of the set of attracting fixed points of proximal elements of $\mathrm{Aut}(\Omega)$.

By work of Vey [19, Prop. 3] and Benoist [2, Lem. 3.6.ii], the proximal limit set is also

- the closure of the set of extremal points of $\overline{\Omega}$;
- the closure of the set of attracting fixed points of proximal elements of Γ, for any cocompact closed subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$;
- the smallest (for inclusion) closed Γ-invariant non-empty subset of $\mathrm{P}(V)$ for any cocompact closed subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$.

If Ω is an ellipsoid, i.e. a rank-one symmetric divisible convex set, then $\Lambda_\Omega^{\mathrm{prox}} = \partial\Omega$ and $\mathrm{Aut}(\Omega)$ acts transitively on it. If Ω is a higher-rank symmetric irreducible divisible convex set, then $\Lambda_\Gamma^{\mathrm{prox}}$ is an analytic submanifold of $\mathrm{P}(V)$ of dimension less than $\dim(V) - 2$, and hence is a proper subset of $\partial\Omega$ (see [10, §7]), on which $\mathrm{Aut}(\Omega)$ acts transitively.

Our goal is to prove the following result.

Theorem 1.3. — *Let $\Omega \subset \mathrm{P}(V)$ be a rank-one divisible convex set. Then $\Lambda_\Omega^{\mathrm{prox}} = \partial\Omega$.*

Combined with Zimmer's higher-rank rigidity theorem [21, Th. 1.4], Theorem 1.3 yields the following answer to a question of Benoist [7, Prob. 5].

Corollary 1.4. — *Let $\Omega \subset \mathrm{P}(V)$ be a non-symmetric irreducible divisible convex set. Then $\Lambda_\Omega^{\mathrm{prox}} = \partial\Omega$.*

Let Ω be a rank-one divisible convex set. The conclusion of Theorem 1.3 holds trivially if Ω is symmetric (i.e. is an ellipsoid). Thus we may assume that Ω is not symmetric and, hence, that $\mathrm{Aut}(\Omega)$ is discrete and Zariski-dense in $\mathrm{PGL}(V)$ (and finitely generated).

Benoist [4, Th. 1.1] proved that $\mathrm{Aut}(\Omega)$ is Gromov-hyperbolic if and only if Ω is *strictly convex* (i.e. all points of $\partial\Omega$ are extremal), if and only if $\partial\Omega$ is $\mathcal{C}^1$. In this case, strict convexity implies that $\Lambda_\Omega^{\mathrm{prox}} = \partial\Omega$ (since $\Lambda_\Omega^{\mathrm{prox}}$ is the closure of the set of extremal points). One may find more precise results on the regularity of $\partial\Omega$ in [4].

Benoist [5] also studied non-strictly convex three-dimensional rank-one divisible convex sets. He constructed examples and established a precise description of these that implies that $\Lambda_\Omega^{\mathrm{prox}} = \partial\Omega$.

Islam–Zimmer [15] generalised Benoist's description to higher-dimensional rank-one divisible convex sets, under the assumption that $\mathrm{Aut}(\Omega)$ is relatively hyperbolic, and their result implies that $\Lambda_\Omega^{\mathrm{prox}} = \partial\Omega$ in this case. M. Bobb [11] also generalised Benoist's result under the assumption that each non-trivial face of Ω (see Section 2.2) is contained in a properly embedded simplex of dimension $\dim(V) - 2$, namely a closed simplex $S \subset \overline{\Omega}$ whose relative interior (see Section 2.2) is exactly $S \cap \Omega$; Bobb's result also implies that $\Lambda_\Omega^{\mathrm{prox}} = \partial\Omega$.

1.3. Organisation of the paper. — In Section 2, we recall basic notions of projective geometry. In particular, we recall the definition of the Hilbert metric on Ω and how it naturally extends to the projective closure $\overline{\Omega}$.

In Section 3, we establish a weak, convex projective version (Lemma 3.1) of Sullivan's celebrated shadow lemma. This result can be seen as a consequence of a more standard convex projective version of the Sullivan shadow lemma proved in [9, Lem. 4.2], where we develop the theory of Patterson–Sullivan densities in convex projective geometry.

In Section 4, we establish two topological results (Lemmas 4.1 and 4.3), which concern the arrangement of faces on the boundary of a convex set.

In Section 5, we use Sections 3 and 4 to prove Theorem 1.3.

2. Reminders in convex projective geometry

2.1. The Hilbert metric. — In the whole paper we fix a real vector space $V = \mathbb{R}^{d+1}$, where $d \geq 1$. Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set. Recall that Ω admits an $\mathrm{Aut}(\Omega)$-invariant proper metric called the *Hilbert metric* and defined by the following formula: for $(a, x, y, b) \in \partial\Omega \times \Omega \times \Omega \times \partial\Omega$ aligned in this order,

$$d_\Omega(x, y) = \frac{1}{2} \log([a, x, y, b]), \tag{1}$$

where $[a, x, y, b]$ is the cross-ratio of the four points, given by

$$[a, x, y, b] = \frac{\|b - x\| \cdot \|a - y\|}{\|a - x\| \cdot \|b - y\|}, \tag{2}$$

where $\| \cdot \|$ is a norm on affine chart of $\mathrm{P}(V)$ containing $\overline{\Omega}$.

If Ω is an ellipsoid, then (Ω, d_Ω) is the Klein model of the real hyperbolic space of dimension d. If Ω is a d-simplex, then (Ω, d_Ω) is isometric to $\mathbb{R}^d$ endowed with a norm, with hexagonal balls when $d = 2$; see [18, Prop. 1.7] or [13].

2.2. Faces of the boundary. — Let us recall some basic notions about convexity. For any topological space X and any subspace Y, we denote by $\mathrm{int}_X(Y)$ (resp. $\partial_X Y$) the interior (resp. boundary) of Y with respect to X; if $X = \mathrm{P}(V)$, then we just write $\mathrm{int}\, Y := \mathrm{int}_{\mathrm{P}(V)} Y$ (resp. $\partial Y := \partial_X Y$) and call it the interior (resp. boundary) of Y. Let $K \subset \mathrm{P}(V)$ be properly convex, i.e. convex and bounded in some affine chart.

- The *relative interior* (resp. *relative boundary*) of K, denoted by $\mathrm{int}_{\mathrm{rel}}(K)$ (resp. $\partial_{\mathrm{rel}} K$) is its topological interior (resp. boundary) with respect to the projective subspace it spans.
- For $x \in \overline{K}$, the *open face* of x in $\overline{K}$, denoted by $F_K(x)$, consists of the points $y \in \overline{K}$ such that $[x, y]$ is contained in the relative interior of a (possibly trivial) segment contained in $\overline{K}$. The *closed face* of x is $\overline{F}_K(x) := \overline{F_K(x)}$.
- A point $x \in \overline{K}$ is said to be *extremal* (resp. *strongly extremal*) if $F_K(x) = \{x\}$ (resp. $F_K(x) = \{x\}$ and $[x, y] \cap \mathrm{int}_{\mathrm{rel}} K \neq \emptyset$ for $y \in \partial_{\mathrm{rel}} K \smallsetminus \{x\}$); one says that K is *strictly convex* if all the points in the relative boundary are extremal (and hence strongly extremal).
- Assume that K spans $\mathrm{P}(V)$ and let $\xi \in \partial K$. A *supporting hyperplane* of K at ξ is a hyperplane that contains ξ but does not intersect $\mathrm{int}(K)$. The existence of such hyperplane is given by the first geometric form of the Hahn–Banach theorem.

2.3. Extension of the Hilbert metric to the projective closure. — We extend the definition of the Hilbert distance d_Ω to pairs of points x, y in the closure $\overline{\Omega}$. If y is in the open face $F_\Omega(x)$ of x, then we set $d_{\overline{\Omega}}(x, y) := d_{F_\Omega(x)}(x, y)$, where $d_{F_\Omega(x)}$ is the Hilbert metric on $F_\Omega(x)$, seen as a properly convex open subset of the projective subspace it spans. If y is not in $F_\Omega(x)$, then we set $d_{\overline{\Omega}}(x, y) = \infty$.

For any $x \in \overline{\Omega}$ and $R > 0$, we denote by $\overline{B}_{\overline{\Omega}}(x, R)$ (resp. $B_{\overline{\Omega}}(x, R)$) the set of points $y \in \overline{\Omega}$ with $d_{\overline{\Omega}}(x, y) \leq R$ (resp. $d_{\overline{\Omega}}(x, y) < R$). The following elementary fact plays an important role in this paper.

FACT 2.1. — *Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set. The function $d_{\overline{\Omega}} : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous. As a consequence, for any $R > 0$, the map*

$$\begin{aligned} \overline{B}_{\overline{\Omega}}(\cdot, R) : \overline{\Omega} &\longrightarrow \{\textit{compact subsets of } \overline{\Omega}\} \\ \xi &\longmapsto \overline{B}_{\overline{\Omega}}(\xi, R) \end{aligned}$$