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GLOBAL EXISTENCE AND ASYMPTOTICS FOR QUASI-LINEAR ONE-DIMENSIONAL KLEIN-GORDON EQUATIONS WITH MILDLY DECAYING CAUCHY DATA

by Annalaura Stingo

Abstract. — Let u be a solution to a quasi-linear Klein-Gordon equation in one-space dimension, $\Box u + u = P(u, \partial_t u, \partial_x u; \partial_t\partial_x u, \partial_x^2 u)$, where P is a homogeneous polynomial of degree three, and with smooth Cauchy data of size $\varepsilon \to 0$. It is known that, under a suitable condition on the nonlinearity, the solution is global-in-time for compactly supported Cauchy data. We prove in this paper that the result holds even when data are not compactly supported but just decaying as $\langle x\rangle^{-1}$ at infinity, combining the method of Klainerman vector fields with a semiclassical normal forms method introduced by Delort. Moreover, we get a one term asymptotic expansion for u when $t \to +\infty$.

Résumé (*Existence globale et comportement asymptotique de petites solutions pour des équation de Klein-Gordon critiques 1D*). — Soit u une solution d'une équation de Klein-Gordon quasi-linéaire en dim. 1 d'espace, $\Box u + u = P(u, \partial_t u, \partial_x u; \partial_t\partial_x u, \partial_x^2 u)$, où P est un polynôme homogène de degré trois, avec données initiales régulières de taille $\varepsilon \to 0$. Il est connu que, sous certaines conditions sur la non-linéarité, la solution est globale en temps pour des données initiales à support compact. Nous montrons que ce résultat est aussi vrai quand les données ne sont pas à support compact mais seulement décroissantes à l'infini comme $\langle x\rangle^{-1}$, en combinant la méthode des champs de vecteurs de Klainerman avec une méthode de formes normales semi-classiques introduite par Delort. De plus, nous obtenons un développement asymptotique à un terme pour u lorsque $t \to +\infty$, prouvant ainsi un résultat de scattering modifié.

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Introduction

The goal of this paper is to prove the global existence and to study the asymptotic behavior of the solution u of the one-dimensional nonlinear Klein-Gordon equation, when initial data are small, smooth and slightly decaying at infinity. We will consider the case of a quasi-linear cubic nonlinearity, namely a homogeneous polynomial P of degree 3 in $(u, \partial_t u, \partial_x u;\, \partial_t\partial_x u, \partial_x^2 u)$, affine in $(\partial_t\partial_x u, \partial_x^2 u)$, so that the initial valued problem is written as

$$(1)\qquad \begin{cases} \square u + u = P(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u) \\ u(1,x) = \varepsilon u_0(x) \\ \partial_t u(1,x) = \varepsilon u_1(x) \end{cases} \qquad t \geq 1, x\in\mathbb{R}, \varepsilon \in \,]0,1[.$$

Our main concern is to obtain results for data which have only mild decay at infinity (i.e., which are $O(|x|^{-1})$, $x \to +\infty$), while most known results for quasi-linear Klein-Gordon equations in dimension 1 are proved for compactly supported data. In order to do so, we have to develop a new approach, that relies on semiclassical analysis, and that allows to obtain for Klein-Gordon equations results of global existence making use of Klainerman vector fields and usual energy estimates, instead of L^2 estimates on the hyperbolic foliation of the interior of the light cone, as done for instance in an early work of Klainerman [23] and more recently in the paper of LeFloch, Ma [26].

We recall first the state of the art of the problem. In general, the problem in dimension 1 is critical, contrary to the problem in higher dimension which is subcritical. In fact, in space dimension d, the best time decay one can expect for the solution is $\|u(t,\cdot)\|_{L^\infty} = O(t^{-\frac{d}{2}})$: therefore, in dimension 1 the decay rate is $t^{-\frac{1}{2}}$, and for a cubic nonlinearity, depending for example only on u, one has $\|P(u)\|_{L^2} \leq Ct^{-1}\|u(t,\cdot)\|_{L^2}$, with a time factor t^{-1} just at limit of integrability. In space dimension $d\geq 3$, it is well known from works of Klainerman [23] and Shatah [33] that the analogous problem has global-in-time solutions if ε is sufficiently small. In [23], Klainerman proved it for smooth, compactly supported initial data, with nonlinearities at least quadratic, using the Lorentz invariant properties of $\square + 1$ to derive uniform decay estimates and generalized energy estimates for solutions u to linear inhomogeneous Klein-Gordon equations. Simultaneously, in [33] Shatah proved this result for smooth and integrable initial data, extending Poincaré's theory of normal forms for ordinary differential equations to the case of nonlinear Klein-Gordon equations. For space dimension $d = 2$, in [16] Hörmander refined Klainerman's techniques to obtain new time decay estimates of solutions to linear inhomogeneous Klein-Gordon equations. He showed that, for quadratic nonlinearities, the solution exists over $[-T_\varepsilon, T_\varepsilon]$ with an existence time T_ε such that $\lim_{\varepsilon\to 0}\varepsilon \log T_\varepsilon = \infty$ (while $\lim_{\varepsilon\to 0}\varepsilon^2 T_\varepsilon = \infty$ for $d = 1$). In addition, he conjectured that $T_\varepsilon = \infty$ (while for $d = 1$, $\liminf_{\varepsilon\to 0}\varepsilon^2 \log T_\varepsilon > 0$). The first conjecture has been proved by Ozawa, Tsutaya and Tsutsumi in [30] in the semi-linear case, after partial results by Georgiev, Popivanov in [10], and Kosecki in [25] (for nonlinearities verifying some "suitable null conditions"). Later, in [31] Ozawa, Tsutaya and Tsutsumi announced the extension of their proof to the quasi-linear case and studied scattering of solutions. In space dimension 1, Moriyama, Tonegawa and Tsutsumi [29] have shown that the solution exists on a time interval of length longer or equal to e^{c/ε^2}, where ε is the Cauchy data's size, with a nonlinearity vanishing at least at order three at zero, or semi-linear. They also proved that the corresponding solution asymptotically approaches the free solution of the Cauchy problem for the linear Klein-Gordon equation. The fact that in general the solution does not exist globally in time was proved by Yordanov in [35], and independently by Keel and Tao [21]. However, there exist examples of nonlinearities for which the corresponding solution is global-in-time: on one hand, if P depends only on u and not on its derivatives; on the other hand, for seven special nonlinearities considered by Moriyama in [28]. A natural question is then posed by Hörmander, in [15, 16]: can we formulate a structure condition for the nonlinearity, analogous to the null condition introduced by Christodoulou [3] and Klainerman [24] for the wave equation, which implies global existence? In [5, 6] Delort proved that, when initial data are compactly supported, one can find a *null condition*, under which global existence is ensured. This condition is likely optimal, in the sense that when the structure hypothesis is violated, he constructed in [4] approximate solutions blowing up at e^{A/ε^2}, for an explicit constant A. This suggests that also the exact solution of the problem blows up in time at e^{A/ε^2}, but this remains still unproven.

Once global existence is ensured, a natural question that arises concerns the long time behavior of the solutions. While for $d \geq 2$ it is known that the global solution behaves like a free solution, in space dimension one, only few results were known, including for the simpler equation

$$\Box u + u = \alpha u^2 + \beta u^3 + \text{order}\, 4.$$

For this equation, Georgiev and Yordanov [11] proved that, when $\alpha = 0$, the distance between the solution u and linear solutions cannot tend to 0 when $t \to \infty$, but they do not obtain an asymptotic description of the solution (except for the particular case of sine-Gordon $\Box u + \sin u = 0$, for which they use methods of "nonlinear scattering"). In [27], Lindblad and Soffer studied the scattering problem for long range nonlinearities, proving that for all prescribed asymptotic solutions there is a solution of the equation with such behavior, for some choice of initial data, and finding the complete asymptotic expansion of the solutions. In [14], a sharp asymptotic behavior of small solutions in the quadratic, semilinear case is proved by Hayashi and Naumkin, without the condition of compact support on initial data, using the method of normal forms of Shatah. The only other cases in dimension one for which the asymptotic behavior is known concern nonlinearities studied by Moriyama in [28], where he showed that solutions have a free asymptotic behavior, assuming the initial data to be sufficiently small and decaying at infinity.

Some results about global existence and long time behavior are also known for solutions to systems of coupled Klein-Gordon equations. In dimension $d = 3$, we cite the work of Germain [12], and of Ionescu, Pausader [18], for a system of coupled Klein-Gordon equations with different speeds, with a quadratic nonlinearity, respectively in the semilinear case for the former, and in the quasi-linear one for the latter. For data small, smooth and localized, they prove that a global solution exists and scatters. In dimension $d = 2$, Delort, Fang and Xue proved in [8] the global existence of solutions for a quasi-linear system of two Klein-Gordon equations, with masses m_1, m_2, $m_1 \neq 2m_2$ and $m_2 \neq 2m_1$, for small, smooth, compactly supported Cauchy data, extending the result proved by Sunagawa in [34] in the semilinear case. Moreover, they proved that the global existence holds true also in the resonant case, e.g., when $m_1 = 2m_2$, and a convenient null condition is satisfied by nonlinearities. The same result in the resonant case is also proved by Katayama, Ozawa [19], and by Kawahara, Sunagawa [20], in which the structural condition imposed on nonlinearities includes the Yukawa type interaction, which was excluded from the *null condition* in the sense of [8]. We should cite also the paper [32] by Schottdorf, where he proved global well-posedness and scattering result in the semilinear case, in dimension 2 and higher, for small H^s data, using the contraction mapping technique in U^2/V^2 based spaces. There are some results also in dimension 1. In [22], Kim shows that the solution to a system of semilinear cubic Klein-Gordon equations, verifying a suitable structure condition, and with small, non compactly supported initial data in some appropriate Sobolev space, is global-in-time and has the optimal decay $t^{-1/2}$, as t tends to infinity. We should also cite the work of Guo, Han and Zhang [13] on the global existence and the long time behavior of the solution to the one dimensional Euler-Poisson system, under weak conditions on the initial data, and of Candy and Lindblad [2], on the one dimensional cubic Dirac equation.

In most of above mentioned papers dealing with the one dimensional scalar problem, two key tools are used: normal forms methods and/or Klainerman vector fields Z. In particular, the latter are useful since they have good properties of commutation with the linear part of the equation, and their action on the nonlinearity $ZP(u)$ may be expressed from u, Zu using Leibniz rule. This allows one to prove easily energy estimates for $Z^k u$, and then to deduce from them L^∞ bounds for u, through Klainerman-Sobolev type inequalities. However, in these papers the global existence is proved assuming small, *compactly supported* initial data. This is related to the fact that the aforementioned authors use in an essential way a change of variable in hyperbolic coordinates, that does not allow for non compactly supported Cauchy data. Our aim is to extend the result of global existence for cubic quasi-linear nonlinearities in the case of small compactly supported Cauchy data of [5, 6], to the more general framework of data with mild polynomial decay. To do that, we will combine the Klainerman vector fields' method with the one introduced by Delort in [7].

In [7], Delort develops a semiclassical normal form method to study global existence for nonlinear hyperbolic equations with small, smooth, decaying Cauchy data, in the critical regime and when the problem does not admit Klainerman vector fields. The strategy employed is to construct, through semiclassical analysis, some *pseudo-differential* operators which commute with the linear part of the considered equation, and which can replace vector fields when combined with a microlocal normal form method. Our aim here is to show that one may combine these ideas together with the use of Klainerman vector fields to obtain, in one dimension, and for nonlinearities satisfying the null condition, global existence and modified scattering.

In our paper, we prove the global existence of the solution u by a *boostrap* argument, namely by showing that we can propagate some suitable *a priori* estimates made on u. We propagate two types of estimates: some energy estimates on u, Zu, and some uniform bounds on u. To prove the propagation of energy estimates is the simplest task. We essentially write an energy inequality for a solution u of the Klein-Gordon equation in the quasi-linear case (the main reference is the book of Hörmander [16], Chapter 7), and then we use the commutation property of the Klainerman vector fields Z with the linear part of the equation to derive an inequality also for Zu. Moreover, Z acts like a derivation on the nonlinearity, so the Leibniz rule holds and we can estimate ZP in term of u, Zu. Injecting *a priori* estimates in energy inequalities and choosing properly all involved constants allow us to obtain the result.

The main difficulty is to prove that the uniform estimates hold and can be propagated. Actually, as mentioned above, the one dimensional Klein-Gordon equation is critical, in the sense that the expected decay for $\|u(t,\cdot)\|^2_{L^\infty}$ is in t^{-1}, so is not integrable. A drawback of that is that one cannot prove energy estimates that would be uniform as time tends to infinity. Consequently, a Klainerman-Sobolev inequality, that would control $\|u(t,\cdot)\|_{L^\infty}$ by $t^{-1/2}$ times the L^2 norms of u, Zu, would not give the expected optimal L^∞-decay of the solution, but only a bound in $t^{-\frac{1}{2}+\sigma}$ for some positive σ, which is useless to close the bootstrap argument. The idea to overcome this difficulty is, following the approach of Delort in [7], to rewrite (1) in semiclassical coordinates, for some new unknown function v. The goal is then to deduce from the PDE satisfied by v an ODE from which one will be able to get a uniform L^∞ bound for v (which is equivalent to the optimal $t^{-1/2}$ L^∞-decay of u). Let us describe our approach for a simple model of Klein-Gordon equation. Denoting by D_t, D_x respectively $\frac{1}{i}\partial_t$, $\frac{1}{i}\partial_x$, we consider the following:

$$(D_t - \sqrt{1+D_x^2})u = \alpha u^3 + \beta|u|^2u + \gamma|u|^2\bar{u} + \delta\bar{u}^3, \tag{2}$$