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Robert Laterveer

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Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

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ALGEBRAIC CYCLES AND FANO THREEFOLDS OF GENUS 10

BY ROBERT LATERVEER

ABSTRACT. — We show that prime Fano threefolds Y of genus 10 have a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. As a consequence, a certain tautological subring of the Chow ring of powers of Y injects into cohomology.

RÉSUMÉ (*Cycles algébriques et solides de Fano de genre 10*). — Soit Y un solide de Fano d'indice 1 et de genre 10. On montre que Y admet une décomposition de Chow–Künneth multiplicative, au sens de Shen–Vial. Il s'ensuit qu'un certain sous-anneau "tautologique" de l'anneau de Chow des puissances de Y s'injecte en cohomologie.

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ROBERT LATERVEER, Institut de Recherche Mathématique Avancée, CNRS – Université de Strasbourg, 7 Rue René Descartes, 67084 Strasbourg CEDEX, FRANCE • *E-mail :* `robert.laterveer@math.unistra.fr`

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1. Introduction

Given a smooth projective variety Y over $\mathbb{C}$, let

$$A^i(Y) := CH^i(Y)_{\mathbb{Q}}$$

denote the Chow groups of Y (i.e. the groups of codimension i algebraic cycles on Y with $\mathbb{Q}$-coefficients, modulo rational equivalence). The intersection product defines a ring structure on $A^*(Y) = \bigoplus_i A^i(Y)$, the Chow ring of Y [11]. In the case of K3 surfaces, this ring has a remarkable property:

Theorem 1.1 (Beauville–Voisin [3]). — *Let S be a K3 surface. The $\mathbb{Q}$-subalgebra*

$$\big\langle A^1(S), c_j(S) \big\rangle \subset A^*(S)$$

injects into cohomology under the cycle class map.

The Chow ring of abelian varieties also exhibits particular behaviour: there is a multiplicative splitting [1]. Motivated by the cases of K3 surfaces and abelian varieties, Beauville [2] has conjectured that for certain special varieties, the Chow ring should admit a multiplicative splitting (and a certain subring should inject into cohomology). To make concrete sense of Beauville's elusive "splitting property conjecture", Shen–Vial [47] have introduced the concept of *multiplicative Chow–Künneth decomposition*; we will abbreviate this to "MCK decomposition" (for the precise definition, cf. Section 3 below).

It is something of a challenge to understand precisely which varieties admit an MCK decomposition. To give an idea of what is known: hyperelliptic curves have an MCK decomposition [47, Example 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [9, Example 2.3]; K3 surfaces have an MCK decomposition, but certain high degree surfaces in $\mathbb{P}^3$ do not have an MCK decomposition (cf. the examples given in [41]). In this note, we will focus on Fano threefolds and ask the following question:

Question 1.2. — *Let X be a Fano threefold with Picard number 1. Does X admit an MCK decomposition?*

The restriction on the Picard number is necessary to rule out a counterexample of Beauville [2, Examples 9.1.5]. The answer to Question 1.2 is affirmative for cubic threefolds [6], [9], for intersections of two quadrics [30], for intersections of a quadric and a cubic [31] and for prime Fano threefolds of genus 8 [29].

The main result of this note answers Question 1.2 for one more family:

Theorem (= Theorem 5.1). — *Let Y be a prime Fano threefold of genus 10. Then Y has a multiplicative Chow–Künneth decomposition.*

The argument proving Theorem 5.1 is based on the connections between Y and a certain genus 2 curve, and between Y and an index 2 Fano threefold Z (cf. Theorem 2.2). The work of Kuznetsov [20], [21], [23], building these connections on a categorical level inside the set-up of *homological projective duality*, allows us to establish the instances of the *Franchetta property* that are needed to prove the theorem.

Reaping the fruits of Theorem 5.1, we obtain a result concerning the *tautological ring*, which is a certain subring of the Chow ring of powers of Y:

Corollary (= Corollary 7.1). — *Let Y be a prime Fano threefold of genus 10, and $m \in \mathbb{N}$. Let*

$$R^*(Y^m) := \big\langle (p_i)^*(h), (p_{ij})^*(\Delta_Y) \big\rangle \subset A^*(Y^m)$$

be the $\mathbb{Q}$-subalgebra generated by pullbacks of the polarization $h \in A^1(Y)$ and pullbacks of the diagonal $\Delta_Y \in A^3(Y \times Y)$. The cycle class map induces injections

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \quad \textit{for all } m \in \mathbb{N}\,.$$

This is the kind of injectivity result that motivated Beauville's work on the "splitting property conjecture" [2]. To paraphrase Corollary 7.1, one could say that genus 10 Fano threefolds behave like hyperelliptic curves from the point of view of intersection theory (cf. Remark 7.2 below).

Conventions. — In this article, the word *variety* will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A *subvariety* is a (possibly reducible) reduced subscheme that is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(Y)$ the Chow group of j-dimensional cycles on Y with $\mathbb{Q}$-coefficients; for Y smooth of dimension n the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notation $A^j_{hom}(Y)$ will be used to indicate the subgroup of homologically trivial cycles. For a morphism $f\colon X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f.

The contravariant category of Chow motives (i.e. pure motives with respect to rational equivalence as in [46], [40]) will be denoted $\mathcal{M}_{\rm rat}$.

2. Prime Fano threefolds of genus 10

The classification of Fano threefolds is one of the glories of twentieth century algebraic geometry [17]. Fano threefolds that are *prime* (i.e. with Picard group of rank 1 generated by the canonical divisor) come in 10 explicitly described families. In this paper, we will be concerned with one of these families: