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COXETER POLYTOPES AND BENJAMINI–SCHRAMM CONVERGENCE

by Jean Raimbault

Abstract. — We observe that a large part of the volume of a hyperbolic polyhedron is taken by a tubular neighbourhood of its boundary and use this to give a new proof for the finiteness of arithmetic maximal reflection groups following a recent work with M. Frączyk and S. Hurtado. We also study in more depth the case of polygons in the hyperbolic plane.

Résumé (*Polytopes de Coxeter et convergence de Benjamini–Schramm*). — En partant de l'observation qu'au moins une proportion fixée du volume d'un polytope hyperbolique est concentrée dans un voisinage tubulaire de son bord, nous donnons une nouvelle démonstration de la finitude des groupes de réflexion arithmétiques, à la suite d'un travail en commun avec M. Frączyk and S. Hurtado. Nous effectuons aussi une étude plus poussée de ce phénomène dans le cas des polygones du plan hyperbolique.

Let X be a space of constant curvature, which is either a hyperbolic space $\mathbb{H}^d$, a Euclidean space $\mathbb{R}^d$ or a sphere $\mathbb{S}^d$. An hyperplane in X is a one-lower-dimensional complete totally geodesic subspace, and a polytope is a bounded (or in the case of $\mathbb{H}^d$, finite-volume) region delimited by a finite number of hyperplanes. A polytope in X is said to be Coxeter if the dihedral angles between its faces are each of the form π/m for some $m \geq 2$. For Coxeter polytopes in $\mathbb{R}^d$ or $\mathbb{S}^d$ there is a well-known, complete and very intelligible classification of Coxeter polytopes by Coxeter diagrams. On the other hand,

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Coxeter polytopes in $\mathbb{H}^d$ have a very different behaviour and are still quite mysterious. In the sequel, we will thus only be concerned with $X = \mathbb{H}^d$. We will study Coxeter polytopes from a metric viewpoint and establish some results about their shapes when the volume tends to infinity, especially when $d = 2$. A general survey on Coxeter groups in hyperbolic space is given in [13]; a more recent one is given in [3], which focuses on arithmetic aspects.

Let P be a Coxeter polytope in $\mathbb{H}^d$ and let Γ_P be the subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ generated by reflections in the faces of P. This is a discrete subgroup acting on X with fundamental domain P, by the Poincaré polyhedron theorem. Moreover, P is identified with the $\mathbb{H}^d$-orbifold given by $\Gamma_P \backslash \mathbb{H}^d$, by endowing each face of P with the local orbifold structure given by its pointwise stabiliser (the group generated by reflections in the maximal faces that contain it). For $R > 0$, the R-thin part of P is given by

$$P_{\leq R} = \{x \in P : \exists \gamma \in \Gamma_P,\, d(x, \gamma x) \leq R/2, \}$$

and it corresponds to the R-thin part of the orbifold. An easy exercise shows that $P_{\leq R}$ is equal to the set of points in P that are at distance at most $R/2$ from the boundary. The first result in this note is the following, which is essentially a consequence of the hyperbolic isoperimetric inequality as we prove in 1 below.

Theorem 1. — *For every $d \geq 2$, there exists a constant $C(d)$ such that for every Coxeter polytope P of finite volume in $\mathbb{H}^d$, we have*

$$\mathrm{vol}(P_{\leq 2}) \geq C(d)\,\mathrm{vol}(P).$$

In a joint work with M. Frączyk and S. Hurtado [6] it was proven that $\mathrm{vol}(M_{\leq R}) = o(\mathrm{vol}\, M)$ uniformly for M a congruence arithmetic orbifold quotient of a given symmetric space. From this together with Theorem 1 we can quite easily deduce the following result, which was originally proved by Nikulin and Agol–Belolipetsky–Storm–Whyte ([11, 2], respectively). In fact, the inspiration for this note was provided by a recent work of Fisher–Hurtado [5], where they use a lower-level part of [6][1] to give a new proof of Nikulin and Agol–Belolipetsky–Storm–Whyte's result.

Corollary 2. — *For any d there are at most finitely many arithmetic maximal reflection groups in* $\mathrm{PO}(d, 1)$.

Proof. — Let Γ_{P_n} be a sequence of pairwise non-conjugated maximal Coxeter arithmetic lattices, that is P_n is a finite-volume Coxeter polyhedron in $\mathbb{H}^d$ and $\mathrm{vol}(P_n) \to +\infty$. Let Γ_n be the congruence closure of Γ_{P_n}. Theorem D in [6] states that if $\mathrm{vol}(\Gamma_n \backslash \mathbb{H}^d) \to +\infty$ then, putting $M_n = \Gamma_n \backslash \mathbb{H}^d$, we have that for any $R > 0$, $\lim_{n \to +\infty} \frac{\mathrm{vol}(M_n)_{\leq R}}{\mathrm{vol}\, M_n} = 0$. Since P_n is a finite (orbifold) cover of M_n

1. Namely the "arithmetic Margulis lemma", Theorem 3.1 in loc. cit., which is an essential ingredient in the proof of Theorem D in loc. cit.

and the ratio $\frac{\mathrm{vol}(\cdot)_{\leq R}}{\mathrm{vol}}$ is decreasing in finite covers, this would contradict the fact that $\mathrm{vol}(P_n)_{\leq 2} \geq C(d)\,\mathrm{vol}(P_n)$ if we knew that $\mathrm{vol}(\Gamma_n \backslash \mathbb{H}^d) \to +\infty$.

So, to deduce the corollary we need only prove that $\mathrm{vol}(\Gamma_n \backslash \mathbb{H}^d) \to +\infty$. This is the case if the adjoint trace fields of Γ_{P_n} are not of bounded degree. On the other hand, if these trace fields have bounded degree then by [2, Lemma 6.2] and standard arguments (see, e.g. Lemma 5.4 in loc. cit.) we have that $[\Gamma_n : \Gamma_{P_n}]$ is bounded, so that $\mathrm{vol}(\Gamma_n \backslash \mathbb{H}^d) \gg \mathrm{vol}(P_n)$ goes to infinity. □

It is also well known that cofinite reflection groups cannot exist in large dimensions [12, 8], so we may as well say that there are only finitely many congruence (or maximal arithmetic) hyperbolic reflection groups.

In the case of polygons in $\mathbb{H}^2$, we can say more. Let $G = \mathrm{Isom}(\mathbb{H}^d) = \mathrm{PO}(d, 1)$ and let μ_P be the G-invariant measure on the Chabauty space Sub_G of G supported on the conjugacy class of Γ_P. We will use the notion of Benjamini–Schramm convergence introduced in [1, Sections 2-3]; this is the notion of convergence induced by the topology of weak convergence of measures on Sub_G. In this language, Theorem 1 implies that the trivial subgroup is not a limit point of the measures μ_P. When $d = 2$ we can prove the following result, which is much more precise than Theorem 1.

Theorem 3. — *If μ belongs to the closure (in topology of weak convergence of measures on Sub_G) of the set of all μ_P for P a finite-volume Coxeter polygon in $\mathbb{H}^2$, then μ-almost every subgroup is non-trivial and generated by reflections.*

This seems likely to be true in higher dimensions as well, though our very elementary proof does not seem to immediately extend to this setting. Finally, note that any subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ generated by reflections is, in fact, generated by the reflections in the side of a Coxeter polytope (possibly with infinitely many faces), which is a well-known fact[2]

Organisation. — The proof of Theorem 1 is very short and given in Section 1. The rest of the article is dedicated to the proof of Theorem 3; first we collect a few useful facts on the geometry of hyperbolic Coxeter polygons in Section 2, and use them in Section 3 to prove that a Benjamini–Schramm limit of Coxeter polygons is almost surely non-trivial. Independently, we prove in Section 4 that the set of groups generated by reflections is closed in the Chabauty topology and deduce that a Benjamini–Schramm limit of Coxeter polygons is almost surely generated by reflections.

1. Proof of Theorem 1

Fix $d \geq 2$. Let P be a Coxeter polytope in $\mathbb{H}^d$. We will denote by $F_i, i \in I$ the $(d-1)$-faces of P.

2. This is stated at the beginning of [13], and the proof is more or less obvious.

Now let

$$U = \{x \in P : \ d(x, \partial P) \leq 1\}$$

and

$$P' = \{x \in P : \ d(x, \partial P) \geq 1\}.$$

We have $U \subset P_{\leq 2}$ since every point of U is moved by at most 2 by a reflection in a face of P (as previously noted, it is clear that, in fact, $U = P_{\leq 2}$).

For $x \in F_i$, let ν_x be the vector normal to F_i pointing inside P; note that $d(F_i, \exp_x(\nu_x)) = 1$, for all $x \in F_i$. Let $W_i \subset F_i$ defined by

$$W_i = \{x \in F_i : \exp_x(\nu_x) \in P', \, \forall \, j \neq i \, d(F_j, \exp_x(\nu_x)) > 1\}$$

and let

$$F_i' = \{\exp_x(\nu_x) : \ x \in W_i\}.$$

Then the F_i' are disjoint open subsets of $\partial P'$, and their complement $S = \partial P' \setminus \bigcup_{i \in I} F_i'$ is of measure 0 (with respect to the $(d-1)$-dimensional measure on $\partial P'$) as it is equal to the set of points in $\partial P'$ at distance 1 from at least two of the F_i.

For $y \in F_i'$, let ν_y' be the vector orthogonal to F_i' pointing outside P'; then the map

$$E : [0,1] \times \partial P' \setminus S \to \mathbb{H}^d, \ (t, y) \mapsto \exp_y(t\nu_y') \tag{1}$$

has its image inside U. Since the local geometry of the submanifolds F_i' of $\mathbb{H}^d$ depends only on d, we see that the Jacobian of E is uniformly bounded away from 0 (we prove this in detail in 1.1 at the end of the section); let $\varepsilon(d) > 0$ be a lower bound. Moreover, the sets $E([0,1] \times F_i')$ are pairwise disjoint (since a point in $E([0,1] \times F_i')$ is at distance ≤ 1 from exactly one face of ∂P, which is F_i, and at distance > 1 of all others). It follows that

$$\mathrm{vol}_d \left(E([0,1] \times \partial P' \setminus S)\right) \geq \varepsilon(d) \cdot 1 \cdot \mathrm{vol}_{d-1}(\partial P' \setminus S) = \varepsilon(d) \, \mathrm{vol}_{d-1}(\partial P'),$$

so that $\mathrm{vol}(U) \geq \varepsilon(d) \, \mathrm{vol}_{d-1}(\partial P')$. We finish the proof of the theorem with the following chain of inequalities:

$$\begin{aligned} \mathrm{vol}_d(P) &= \mathrm{vol}_d(U) + \mathrm{vol}_d(P') \\ &\leq \mathrm{vol}_d(U) + \mathrm{vol}_{d-1}(\partial P') \\ &\leq \left(1 + \varepsilon(d)^{-1}\right) \mathrm{vol}_d(U) \\ &\leq C(d) \, \mathrm{vol}_d(P_{\leq 2}). \end{aligned}$$

where the second inequality follows from the isoperimetric inequality for hyperbolic space [7, Proposition 6.6], which implies that $\mathrm{vol}_d(P') \leq \mathrm{vol}_{d-1}(\partial P')$.

1.1. Exponential map on equidistant sets. — Let H be a geodesic hyperplane in $\mathbb{H}^d$ and H' a connected component of $\{x \in \mathbb{H}^d : d(x,H) = 1\}$. Let $E : [0,1] \times H' \to \mathbb{H}^d$ be the map defined as in (1). Since all hyperplanes and their equidistant sets are related by isometries, and exponential maps are equivariant with respect to those, our claim will follow if we prove that the Jacobian $\det(DE(x,t))$ is uniformly bounded away from 0 for $x \in H', t \in [0,1]$.

The group $\mathrm{Isom}(H)$ acts transitively on H', and the map E is equivariant with respect to this action. It follows that we need only to prove that $\det(DE(t,x))$ is uniformly bounded away from 0 for a fixed x and $t \in [0,1]$. This is immediate by compacity, since $DE(t,x)$ is invertible for all $t \in \mathbb{R}$.

2. Lemmas on Coxeter polygons

We collect here some preliminary facts about Coxeter polygons in $\mathbb{H}^2$, and give complete proofs for all of them; though they are likely well known it seems more convenient to give their (short) proofs than locate sufficiently precise references for them. First we have a consequence of the collar/Margulis lemma.

Lemma 2.1. — *There exists $\eta > 0$ such that if P is a Coxeter polygon in $\mathbb{H}^2$ then:*

1. *if an edge of P has length $\le \eta$, then its adjacent angles are right angles;*
2. *no two adjacent edges of P have both length $\le \eta$;*
3. *any two non-consecutive vertices of P are at distance at least η.*

Proof. — Let $\Gamma = \Gamma_P$ be the discrete subgroup generated by the reflections σ_e in the sides e of P. Let δ be the constant given by the collar/Margulis lemma for $\mathbb{H}^2$, so that for any $x \in \mathbb{H}^2$, we have that

$$\Gamma_x := \langle \gamma \in \Gamma : d(x, \gamma x) \le \delta \rangle$$

is virtually cyclic. So if e_2 is an edge of P with length $\le \delta/2$ and e_1, e_3 the adjacent edges of P, then the subgroup generated by σ_{e_i} is virtually cyclic (as each of σ_{e_i} moves any vertex of e of less than δ). On the other hand, discrete virtually cyclic subgroups of $\mathrm{PGL}_2(\mathbb{R})$ cannot contain an element of finite order other than 2 (such a subgroup contains an hyperbolic isometry and any finite-order element in the group must preserve the two endpoints of its axis), and as this subgroup contains the rotations about the vertices of e_2, the angles between e_1, e_1 and e_2, e_3 must be right angles. This proves part 1 for any $\eta \le \delta/2$.

We now prove part 2 for any $\eta \le \delta/4$. Assume that there are two consecutive edges with lengths less than η, i.e. three consecutive vertices x_1, x_2, x_3 such that $d(x_1, x_2)$, $d(x_2, x_3) \le \eta$. Then if $\sigma_i, 1 \le i \le 4$ are the reflections in the sides containing the x_j, each x_j is at distance less than 2η of the axis of each σ_i so $d(x_j, \sigma_i x_j) < 4\eta \le \delta$. By the Collar Lemma it follows that the subgroup