ISSN 0012-9593

quatrième série - tome 47 fascicule 2 mars-avril 2014

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 47, 2014, p. 449 à 467

SOLUTION ALGEBRAS OF DIFFERENTIAL EQUATIONS AND QUASI-HOMOGENEOUS VARIETIES: A NEW DIFFERENTIAL GALOIS CORRESPONDENCE

BY Yves ANDRÉ

ABSTRACT. – We develop a new connection between Differential Algebra and Geometric Invariant Theory, based on an anti-equivalence of categories between *solution algebras* associated to a linear differential equation (i.e., differential algebras generated by finitely many polynomials in a fundamental set of solutions), and *affine quasi-homogeneous varieties* (over the constant field) for the differential Galois group of the equation.

Solution algebras can be associated to any connection over a smooth affine variety. It turns out that the spectrum of a solution algebra is an algebraic fiber space over the base variety, with quasi-homogeneous fiber. We discuss the relevance of this result to Transcendental Number Theory.

RÉSUMÉ. – Nous tissons un lien nouveau entre algèbre différentielle et théorie géométrique des invariants, basé sur une anti-équivalence de catégories entre algèbres de solutions associées à une équation différentielle linéaire (i.e., algèbres différentielles engendrées par un nombre fini d'expressions polynomiales en les solutions), et variétés quasi-homogènes affines sur le corps de constantes, pour l'action du groupe de Galois différentiel de l'équation.

On peut associer des algèbres de solutions à toute connexion sur une base affine lisse. Il s'avère que leurs spectres sont toujours des fibrés algébriques sur la base, de fibre quasi-homogène. Nous soulignons le rôle de ce résultat en théorie des nombres transcendants.

Introduction

Let K be a field endowed with a non-zero derivation ∂, with algebraically closed constant field $C = \operatorname{Ker} \partial$. Let

$$\phi(y) = \partial^n y + a_{n-1}\partial^{n-1} y + \cdots + a_0 y = 0$$

be a linear differential equation with coefficients a_i in K, and let $y_0, \ldots, y_{n-1}$ form a C-basis of solutions in some differential extension of K with constant field C.

The Picard-Vessiot algebra of ϕ is the K-algebra generated by the derivatives $\partial^j y_i$ and the inverse of the Wronskian $\det(\partial^j y_i)$. It is the ring of coordinates of a principal homogeneous space over K under the differential Galois group G of ϕ. Through Kolchin's work, this fact has been a source of motivation and applications in the early development of the theory of linear algebraic groups and their principal homogeneous spaces (cf. [8, chap. VIII]).

In this paper, we study the finitely generated differential subalgebras of a Picard-Vessiot algebra, which we call *solution algebras*.

Curiously, traditional differential Galois theory has little to say about solution algebras beyond the Picard-Vessiot case—for instance about the algebraic relations between a single solution y_0 and its derivatives (a problem which occurs in transcendental number theory, cf. 1.7[(1)]).

The traditional differential Galois correspondence classifies differential *subfields* of the *fraction field* of the Picard-Vessiot algebra. No such classification in terms of subgroups of the differential group G exists at the level of differential subalgebras.

For instance, the Picard-Vessiot algebra $C(z)$-algebra R' of the Airy equation $\frac{d^2y}{dz^2} = zy$ is the coordinate ring of SL_2, and the subalgebra A generated by the logarithmic derivative of a single non-zero solution y_0 is a finitely generated differential subalgebra of the fraction field $Q(R')$ (not of R'); the fraction field of A corresponds to a Borel subgroup B of SL_2: one has $Q(R')^B = Q(A)$; but $(R')^B = C$, not A.

As we shall see, the study of solution algebras involves finer notions from geometric invariant theory than just algebraic groups and torsors: in fact, the whole theory of affine quasi-homogeneous varieties comes into play.

The differential Galois correspondence can be restored at the level of solution algebras in the form of an *anti-equivalence of categories between solution algebras as above and affine quasi-homogeneous G-varieties over C*.

After pioneering work by Grosshans, Luna, Popov, Vinberg and others in the seventies, the study of *quasi-homogeneous G-varieties*, i.e., algebraic G-varieties with a dense G-orbit, has now become a rich and deep theory. The precise dictionary given below between the theory of affine quasi-homogeneous varieties and differential Galois theory should thus enrich considerably the latter, and may provide a source of motivation and applications for the former. We take advantage of this correspondence to study the algebraic structure of solution algebras (for instance, linear relations between solutions), with an eye towards transcendental number theory.

1. Statement of the main results

Our results take place in the general context of modules with connection over an affine basis[(2)], but in this introduction, we restrict ourselves to the context of differential modules over a differential ring (in the classical sense).

(1) After completion of this work, D. Bertrand pointed out to us the paper [10], in which this problem is studied for generalized confluent hypergeometric differential equations (cf. also [9]).
(2) For a more geometric setting, see 6.5 (2).

1.1. Picard-Vessiot fields (reminder, cf. [15, 19])

Let (K, ∂) be a differential field with algebraically closed constant field $C = K^\partial$ of characteristic 0. Let $K\langle\partial\rangle$ denote the corresponding ring of differential operators. Let M be a differential module over K, that is, a $K\langle\partial\rangle$-module of finite dimension n over K (for instance $M = K\langle\partial\rangle/K\langle\partial\rangle\phi$, where ϕ is a differential operator as above). The finite direct sums of tensor products $M^{\otimes i} \otimes (M^\vee)^{\otimes j}$ and their subquotient differential modules form a Tannakian category $\langle M\rangle^\otimes$ over C.

A Picard-Vessiot field K' for M is a differential field extension of K with constant field C, in which M and its dual $M^\vee$ are solvable (i.e., $\mathrm{Sol}(M, K') := \mathrm{Hom}_{K\langle\partial\rangle}(M, K')$ and $\mathrm{Sol}(M^\vee, K')$ have dimension n over C), and which is minimal for this property. Such a differential field exists and is unique up to non-unique isomorphism. The differential Galois group of M,

$$G = \mathrm{Aut}_\partial K'/K,$$

is a linear algebraic group over C which acts faithfully on $\mathrm{Sol}(M, K')$.

The differential Galois correspondence is an order-reversing bijection between intermediate differential extensions $K \subset L \subset K'$ and closed subgroups $H < G$, given by $H = \mathrm{Aut}_\partial K'/L$ and $L = (K')^H$. One has $\mathrm{tr.deg}_K L = \dim G - \dim H$.

1.2. Solution fields

1.2.1. Definition. – A *solution field* (L, ∂) for M is a differential field extension of (K, ∂) with constant field $L^\partial = C$, which is generated by the image of a $K\langle\partial\rangle$-morphism $v : M \to L$.

For instance, the Picard-Vessiot field K' is a solution field for $M^n \oplus (M^\vee)^n$.

In the next theorem, "solution field" means "solution field for some $N \in \langle M\rangle^\otimes$".

1.2.2. Theorem. –

1. *Any solution field L embeds as a differential subfield of the Picard-Vessiot field K'.*
2. *Conversely, an intermediate differential field $K \subset L \subset K'$ is a solution field if and only if the corresponding subgroup $H < G$ is* observable *(i.e., G/H is quasi-affine). In fact, H is the isotropy group of any solution $v \in \mathrm{Sol}(N, K')$ whose image generates L.*
3. *For any solution field $L = (K')^H$, $\mathrm{Aut}_\partial L/K = N_G(H)/H$.*

1.3. Picard-Vessiot algebras

Even though this result is formulated in terms of traditional differential Galois theory of differential fields, our proof uses the generalized differential Galois theory for differential rings developed in [3] (working over differential rings rather than fields is natural, useful, and sometimes necessary in some contexts).

Let (R, ∂) be a differential ring with constant field C. We assume that (R, ∂) is simple, i.e., has no non-zero proper differential ideal. It is then known that R is an integral domain, and we denote by K its quotient field.

Let M be a differential module of finite type over R. It can be shown that M is projective, and so are all the finite direct sums of tensor products $M^{\otimes i} \otimes (M^\vee)^{\otimes j}$ and their subquotient differential modules, which form a Tannakian category $\langle M\rangle^\otimes$ over C (equivalent to $\langle M_K\rangle^\otimes$), cf. 2.2.1 below (instead of $M^{\otimes i}\otimes(M^\vee)^{\otimes j}$, one may consider $M^{\otimes i}\otimes(\det M)^{\otimes -j}$, where $\det M$ denotes the top exterior power).

The Picard-Vessiot algebra R' for M is the R-subalgebra of the Picard-Vessiot field K' for M_K generated by $\langle M, \mathrm{Sol}(M,K')\rangle$ and $\langle M^\vee, \mathrm{Sol}(M^\vee,K')\rangle$, its spectrum is a torsor under G_R, and $G = \mathrm{Aut}_\partial(R'/R)$.

1.4. Solution algebras

1.4.1. Definition. – A *solution algebra* (S,∂) for M is a differential R-algebra without zero-divisor, whose quotient field has constant field C, and which is generated by the image of a $R\langle\partial\rangle$-morphism $v : M \to S$.

The link with the previous definition is the following (cf. 4.2.2): a differential algebra extension S/R is a solution algebra for M if and only if it is a finitely generated R-algebra without zero-divisor and its quotient field L is a solution field for M_K; any solution field L for M_K is the quotient field of a solution algebra for M.

In the next theorem, "solution algebra" means "solution algebra for some $N \in \langle M\rangle^{\otimes}$".

1.4.2. Theorem. – 1. *Any differential finitely generated sub-R-algebra of the Picard-Vessiot algebra R' is a solution algebra.*

2. *If S is a solution algebra, then for any embedding of the quotient field L of S into K', S is contained in the Picard-Vessiot algebra R'.*
3. *For any solution algebra S generated by a solution v, $\mathrm{Spec}(S_{K'})^\partial$ is the closure $\overline{G.v}$ of the orbit $G.v \subset \mathrm{Sol}(M,K')$. This provides an anti-*equivalence *of categories between solution algebras and affine quasi-homogeneous G-varieties.*
4. *If $H < G$ is observable, $(R')^H$ is a solution algebra if and only if H is Grosshans (i.e., $C[G/H]$ is finitely generated).*
5. *A solution algebra S is simple (as a differential ring) if and only if it is generated by a solution v for which the orbit $G.v$ is closed. In that case, $S = (R')^H$.*
6. *A solution field L is the quotient field of a unique solution algebra S if and only if the image $\bar{H}$ of H in the reductive quotient $\bar{G}$ of G is reductive and $N_{\bar{G}}(\bar{H})/\bar{H}$ is finite. In that case, S is simple.*
7. *Assume that R is finitely generated over C. Then, locally for the étale topology on* $\mathrm{Spec}R$*, the spectrum of a solution algebra S generated by a solution v is isomorphic to $(\overline{G.v})_R$ (in particular, it is an algebraic fiber bundle over* $\mathrm{Spec}R$*).*

1.5. From affine quasi-homogeneous varieties to differential modules

On combining the previous theorem with the constructive solution [16] of inverse differential Galois problem and the triviality of torsors over $C[z]$ under (pull-back of) reductive groups over C [20], one obtains the following

1.5.1. Theorem. – 1. *The differential Galois group G of any semisimple differential module M over $(C[z], \frac{d}{dz})$ is connected reductive, and the spectrum of any solution algebra S for M satisfies* $\mathrm{Spec}\, S \cong Z_{C[z]}$ *for some affine quasi-homogeneous G-variety Z over C.*