ISSN 0012-9593

quatrième série - tome 48 fascicule 1 janvier-février 2015

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition / *Publication*

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Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 48, 2015, p. 1 à 39

CASCADES IN THE DYNAMICS OF MEASURED FOLIATIONS

BY CURTIS T. McMULLEN

ABSTRACT. – This paper studies the behavior of harmonic measured foliations on compact Riemann surfaces. Cascades in the dynamics of such a foliation can occur as its relative periods are varied. We show that in the case of genus 2, the bifurcation locus arising from such a variation is a closed, countable set of $\mathbb{R}$ that embeds in ω^ω.

RÉSUMÉ. – Nous étudions le comportement des feuilletages mesurés harmoniques sur les surfaces de Riemann compactes. Quand les périodes relatives varient, on peut observer des cascades dans la dynamique d'un tel feuilletage. Dans le cas du genre 2, on montre que le lieu de bifurcation résultant d'une telle variation est un sous-ensemble dénombrable et fermé de $\mathbb{R}$, qui se plonge dans ω^ω.

1. Introduction

Let X be a compact Riemann surface of genus g. Every harmonic 1-form ρ on X is the pullback of a linear form under a suitable *period map*

$$\pi : X \to E \cong \mathbb{R}^r/\mathbb{Z}^r.$$

The leaves of the associated measured foliation $\mathcal{F}(\rho)$ of X map under π into the leaves of an irrational foliation of E.

When $r = 2$, the behavior of $\mathcal{F}(\rho)$ is strongly influenced by the *degree* of the period map. For example, if $\mathcal{F}(\rho)$ is periodic then its degree must be zero. At the other extreme, in § 6 we will see:

THEOREM 1.1. – *There is no minimal foliation of degree zero.*

The degree depends only on the *absolute periods* of ρ, given by the real cohomology class $[\rho] \in H^1(X)$. In the case of genus $g = 2$, there is one important remaining invariant, the *relative period* of ρ along a path connecting its zeros (see § 7). In § 8 we will show:

Research supported in part by the NSF and MSRI. Revised in September 2013.

Theorem 1.2. – *Let (X_t, ρ_t) be a family of harmonic forms of genus two with constant absolute periods and relative period t. Assume (X_0, ρ_0) has degree zero. Then the bifurcation locus*

$$B = \{t \, : \, \mathcal{F}(\rho_t) \text{ is not periodic}\}$$

is a closed, countable subset of $\mathbb{R}$ which embeds in ω^ω.

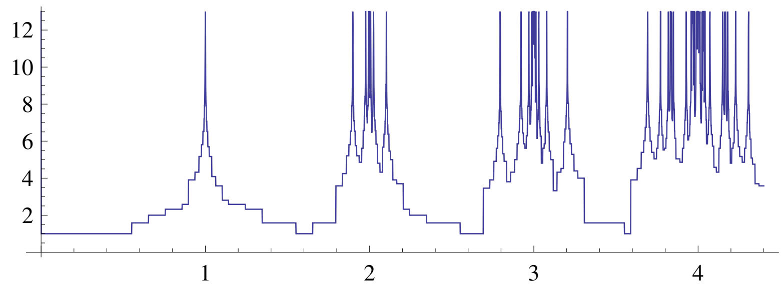

Figure 1. Cascades in the dynamics of $F_t : [0,1] \to [0,1]$.

Example: 1-dimensional dynamics

The spikes in Figure 1 give the bifurcation locus for a family (X_t, ρ_t) that depends on an irrational number $L \in (0,1)$. Here a transversal to $\mathcal{F}(\rho_t)$ determines an interval exchange map

$$F_t : [0,1] \to [0,1].$$

The map F_t rotates $[0,1]$ by t, then rotates the subintervals $[0,L]$ and $[L,1]$ each by $-t$. (Rotation of $[0,a]$ by b is given by $x \mapsto (x+b) \bmod a$.)

The period $N(t)$ of $F_t(x)$ is finite iff $t \notin B$. The figure shows the graph of $y = \log_2 N(t)$ (calculated using Rauzy induction) in the case $L = \pi/7$, whose behavior is typical. Theorem 1.2 shows that structural stability is dense in this and similar families of interval exchange transformations.

Homological invariants

To place these results in a broader setting, consider a general closed 1-form ρ on a compact, oriented n-manifold X. The Poincaré dual of $[\rho] \in H^1(X)$ maps to a natural cycle

$$\text{flux}(X, \rho) \in H_{n-1}(E),$$

which records the average distribution of the leaves of $\mathcal{F}(\rho)$ under the period map $\pi : X \to E$.

The vanishing of flux generalizes the vanishing of degree. As we will see in § 4, zero flux encourages the leaves of $\mathcal{F}(\rho)$ to double back in a manner that is delicate to combine with minimality. Minimal foliations with zero flux in genus three will be discussed in § 5. Such examples cannot exist in genus two, as we will see in § 6 and § 7.

A measurable set $A \subset X$ is *saturated* if it is a union of leaves of $\mathcal{F}(\rho)$. In this case $d(\rho|A) = 0$ as a current. We define the *content* of ρ to be the smallest convex set

$$C(\rho) \subset H^1(X)$$

containing $[\rho|A]$ for every saturated set. In § 2 we show this dynamical invariant is upper semicontinuous, in the sense that

$$\limsup C(\rho_n) \subset C(\rho)$$

whenever $\rho_n \to \rho$.

Similarly, when (X, ρ) has rank $r = n$, we can make sense of the degree (as a real number) for any saturated set, and we define

$$\deg^+(X, \rho) = \sup_A \deg(\pi|A).$$

In the context of Theorem 1.2, in § 8 we will show:

THEOREM 1.3. – *The quantity* $d(t) = \deg^+(X_t, \rho_t) \geq 0$ *is an integer,* B *is the locus where* $d(t) > 0$*, and we have*

$$\limsup_{s \to t} d(s) \leq d(t) - 1 \tag{1.1}$$

whenever $t \in B$.

We emphasize that $d(s)$ is bounded by $d(t) - 1$ (and not just $d(t)$) for all s near t. This evaporation of degree under perturbations is one of the main new phenomena we establish in this paper. The properties of B stated in Theorem 1.2 follow from it immediately, and it plays a key role in the analysis of isoperiodic forms [34].

Quadratic fields

In § 8 we will also show:

THEOREM 1.4. – *If the absolute periods of* ρ_0 *lie in a real quadratic field, then* B *is self-similar about every point.*

This means B is locally invariant under a linear contraction fixing a, for every $a \in B$.

A similar renormalization mechanism causes one bifurcation set to reappear in another. Examples of four such bifurcation sets, with the periods of ρ_0 in $\mathbb{Q}(\sqrt{5})$, are shown in Figure 2. Note that a compressed version of each cascade reappears in the next, near $t = (3 + \sqrt{5})/2 = 2.618\ldots$ Using this idea we give explicit examples where ω^k embeds in B (§ 10). For an example with $B \cong \omega^\omega$, see [34, § 11].

Appendix: Teichmüller curves

Much attention has focused on the family of foliations arising from parallel geodesics with varying slopes on the flat surface $(X, |\omega|)$. These foliations are given by $\rho_t = \mathrm{Re}(e^{it}\omega)$.

The invariants discussed in § 2 can also be studied for such families. In the Appendix we show that the (rapidly fluctuating) convex set $C(\rho_t) \subset H^1(X)$ can be readily determined whenever (X, ω) generates a Teichmüller curve. An example associated to billiards in the regular pentagon is shown in Figure 3. This convex set records the cylinder decomposition at each cusp of $\mathrm{SL}(X, \omega)$, and can be computed by a continued fraction algorithm in $\mathbb{Q}(\sqrt{5})$. For another perspective, we also show:

> *The function graphed in Figure 3 arises from the radial limits* $F : S^1 \to \mathbb{P}\mathcal{ML}_g$ *of a suitable Teichmüller disk* $f : \Delta \to \mathcal{T}_g$.