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PERCOLATION BY CUMULATIVE MERGING AND PHASE TRANSITION FOR THE CONTACT PROCESS ON RANDOM GRAPHS

BY LAURENT MÉNARD AND ARVIND SINGH

ABSTRACT. – Given a weighted graph, we introduce a partition of its vertex set such that the distance between any two clusters is bounded from below by a power of the minimum weight of both clusters. This partition is obtained by recursively merging smaller clusters and cumulating their weights. For several classical random weighted graphs, we show that there exists a phase transition regarding the existence of an infinite cluster.

The motivation for introducing this partition arises from a connection with the contact process as it roughly describes the geometry of the sets where the process survives for a long time. We give a sufficient condition on a graph to ensure that the contact process has a non trivial phase transition in terms of the existence of an infinite cluster. As an application, we prove that the contact process admits a sub-critical phase on random geometric graphs and random Delaunay triangulations.

RÉSUMÉ. – Étant donné un graphe pondéré, nous introduisons une partition de l'ensemble de ses sommets vérifiant la propriété suivante : la distance entre deux parties est inférieure au minimum du poids total de chaque partie élevé à une certaine puissance. Cette partition s'obtient en regroupant successivement des ensembles de sommets et en cumulant leur poids. Pour plusieurs modèles de graphes pondérés aléatoires, nous montrons que l'existence d'une partie infinie présente une transition de phase.

Notre motivation pour l'étude de cette partition provient d'un lien avec le processus de contact et nous donnons une condition suffisante pour la survie du processus en termes d'existence d'une partie infinie. Nous appliquons cette condition pour prouver que le processus de contact sur des graphes géométriques et des triangulations de Delaunay aléatoires admet une transition de phase non triviale.

1. Introduction

The initial motivation of this work is the study of the contact process on an infinite graph with unbounded degrees. The contact process is a classical model of interacting particle system introduced by Harris in [6]. It is commonly seen as a model for the spread of an infection inside a network. Roughly speaking, given a graph G with vertex set V, the contact

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process on G is a continuous time Markov process taking value in $\{0,1\}^V$ (sites having value 1 at a given time are said to be *infected*) and with the following dynamics:

- Each infected site heals at rate 1.
- Each healthy site becomes infected at rate λN where $\lambda > 0$ is the infection parameter of the model and N is the number of infected neighbors.

We give a rigorous definition of the contact process in Section 4 and refer the reader to the books of Liggett [7, 8] for a comprehensive survey on interacting particle systems, including the contact process. Durett's book [5] also provides a nice survey on these models in the setting of random graphs. An important feature of the model is the existence of a critical infection rate λ_c such that the process starting from a finite number of infected sites dies out almost surely when $\lambda < \lambda_c$ but has a positive probability to survive for all times as soon as $\lambda > \lambda_c$.

It is a general result that, on any infinite graph, there exists a super-critical phase, i.e., $\lambda_c < +\infty$ (on a finite graph, the process necessarily dies out since it takes values in a finite space with zero being the unique absorbing state). This follows, for instance, from comparison with an oriented percolation process, see [7]. On the other hand, if the graph has bounded degrees, then there also exists a non trivial sub-critical phase, i.e., $\lambda_c > 0$. This can be seen by coupling the contact process with a continuous time branching random walk with reproduction rate λ. Thus, the phase transition is non degenerated on any vertex-transitive graph such as $\mathbb{Z}^d$ and regular trees. The behavior of the contact process on those graphs has been the topic of extensive studies in the last decades and is now relatively well understood. In particular, depending on the graph, there may exist a second critical value separating a strong and weak survival phase (see [12, 13, 14] for such results on trees). Again, we refer the reader to [5, 7, 8] and the references therein for details.

Comparatively, much less is known about the behavior of the contact process on more irregular graphs. Yet, in the last years, there has been renewed interest in considering these kinds of graphs as they naturally appear as limits of finite random graphs such as Erdős-Rényi graphs, configuration models or preferential attachment graphs [1, 2, 3, 4, 10, 11].

However, without the boundedness assumption on the degree of G, the situation is much more complicated and the existence of a sub-critical phase is not guaranteed. For example, Pemantle [12] proved that, on a Galton-Watson tree with reproduction law B such that, asymptotically, $\mathbf{P}\{B \geq x\} \geq \exp(-x^\beta)$ for some $\beta < 1$, then $\lambda_c = 0$. Thus, the degree distribution of a random tree can have moments of all orders and yet the contact process on it will still survive with positive probability even for arbitrarily small infection rates. This is a very different behavior from the one observed on regular trees with similar average degree and it indicates that the survival of the contact process depends on finer geometric aspects of the underlying graph than just its growth rate. In the case of Galton-Watson trees, we expect the critical value to be positive as soon as the reproduction law has exponential moments. This still remains to be proved, even for progeny distributions with arbitrarily light tails. In fact, to the best of our knowledge, there is no (non-artificial) example of a graph with unbounded degrees for which it has been shown that λ_c is non-zero. Worst, predictions of physicists hinting at non-zero values of λ turned out to be wrong, see for instance [4]. The main goal of this paper is to find a sufficient condition on a graph G for the contact process to have a non-trivial sub-critical phase and then give examples of classical graphs satisfying this condition.

Let us quickly explain the difficulty met when studying the contact process on a graph with unbounded degrees. First, the comparison between the contact process and the branching random walk becomes useless since the later process always survives with positive probability. This follows from the fact that it survives on finite star graphs with large enough degree. Therefore, we must find another way to control the influence of sites with large degree. Those sites should be seen as "sources" which, once infected, will generate many new infections. Some of those infections may, in turn, reach other sites with high degree. This can lead to an amplification effect preventing the process from ever dying depending on the repartition of these sources inside the graph. In this respect, it is not so hard to find conditions for the process to survive: one just has to find groups of vertices containing enough sites with very large degree to make the process super-critical. On the other hand, in order to ensure the death of the process, it is necessary to consider the global geometry of the graph. The following heuristic is meant to shed some light on this last statement and will motivate the introduction of a particular partition of the vertex set of the graph which we call *cumulatively merged partition*. This partition will, ultimately, become the main object of interest of this paper.

Heuristic. – It is well known that the contact process on a star graph of degree d (i.e., a vertex joined to d leaves) has a survival time of exponential order (say, to simplify $\exp(d)$) when the infection parameter λ is larger than some value $\lambda_c(d) > 0$. Now, consider the contact process on an infinite graph G with unbounded degrees, and fix a very small infection rate λ so that there are only very few sites in the graph where the contact process is locally super-critical (those with degree larger than say, d_0).

To see the influence of these vertices with anomalously large degree, imagine that we start the process with a single infected site a having degree $d_a > d_0$. In addition, suppose that in a neighborhood of a, every vertex has degree smaller than d_0. Now run the process while forcing a to stay infected for a time of order $\exp(d_a)$ after which the whole star around site a recovers. By that time, roughly $\exp(d_a)$ infections will have been generated by the star around a. But, inside the neighborhood of a, vertices have small degrees so the process is sub-critical and each infection emitted from a propagates only up to a distance with finite expectation and exponential tail. This tells us that the maximal distance reached by the infections generated from a should be roughly of order d_a.

Now imagine that within distance smaller than d_a from a, there is some other vertex b with degree $d_b > d_0$. Suppose also that d_b is much smaller than the distance between a and b. The previous heuristic applied to b tells us that, in that case, the contact process started from site b has little chance of ever infecting a. Thus, infections generated by a will propagate to b but the converse is false. This means that, while a is infected, infections regularly reach site b but this flux stops when a recovers, then b survives for an additional time $\exp(b)$ without reinfecting a. So, the whole process survives for a time of order $\exp(d_a) + \exp(d_b) \approx \exp(d_a)$.

Consider now the case where there is a vertex c, again at distance less than d_a from a, but this time with degree d_c also larger than the distance between a and c. In that case, infections